C^*-ALGEBRAS ASSOCIATED WITH TEXTILE DYNAMICAL SYSTEMS

KENGO MATSUMOTO

Abstract. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is a finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of a C^*-algebra $\mathcal{A}$ with some conditions. The endomorphisms yield a C^*-algebra $\mathcal{O}_\rho$ from the associated Hilbert C^*-bimodule. In this paper, we will extend the notion of C^*-symbolic dynamical system to C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which consists of two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with certain commutation relations κ between their endomorphisms $\{\rho_\alpha\}_{\alpha \in \Sigma^\rho}$ and $\{\eta_a\}_{a \in \Sigma^\eta}$. C^*-textile dynamical systems yield two-dimensional tilings and C^*-algebras $\mathcal{O}^\kappa_{\rho,\eta}$. We will study the structure of the algebras $\mathcal{O}^\kappa_{\rho,\eta}$ and present its K-theory formulae .

1. Introduction

In [21], the author has introduced a notion of λ-graph system as presentations of subshifts. The λ-graph systems are labeled Bratteli diagram with shift transformation. They yield C^*-algebras so that its K-theory groups are related to topological conjugacy invariants of the underlying symbolic dynamical systems. The class of these C^*-algebras include the Cuntz-Krieger algebras. He has extended the notion of λ-graph system to C^*-*symbolic dynamical system*, which is a generalization of both a λ-graph system and an automorphism of a unital C^*-algebra. It is a finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of a unital C^*-algebra $\mathcal{A}$ such that the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with $\mathcal{A}$. A finite labeled graph $\mathcal{G}$ gives rise to a C^*-symbolic dynamical system $(\mathcal{A}_\mathcal{G}, \rho^\mathcal{G}, \Sigma)$ such that $\mathcal{A} = \mathbb{C}^N$ for some $N \in \mathbb{N}$. A λ-graph system $\mathfrak{L}$ is a generalization of a finite labeled graph and yields a C^*-symbolic dynamical system $(\mathcal{A}_\mathfrak{L}, \rho^\mathfrak{L}, \Sigma)$ such that $\mathcal{A}_\mathfrak{L}$ is $C(\Omega_\mathfrak{L})$ for some compact Hausdorff space $\Omega_\mathfrak{L}$ with $\dim\Omega_\mathfrak{L} = 0$. It also yields a C^*-algebra $\mathcal{O}_\mathfrak{L}$. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ provides a subshift denoted by Λ_ρ over Σ and a Hilbert C^*-right $\mathcal{A}$-module $(\phi_\rho, \mathcal{H}^\rho_\mathcal{A}, \{u_\alpha\}_{\alpha \in \Sigma})$ with an orthogonal finite basis $\{u_\alpha\}_{\alpha \in \Sigma}$ and a unital faithful diagonal left action $\phi_\rho : \mathcal{A} \to L(\mathcal{H}^\rho_\mathcal{A})$. By using general construction of C^*-algebras from Hilbert C^*-bimodules established by M. Pimsner [35], a C^*-algebra denoted by $\mathcal{O}_\rho$ from $(\phi_\rho, \mathcal{H}^\rho_\mathcal{A}, \{u_\alpha\}_{\alpha \in \Sigma})$ has been presented in [24]. We call the algebra $\mathcal{O}_\rho$ *the C^*-symbolic crossed product of $\mathcal{A}$ by the subshift* Λ_ρ. If $\mathcal{A} = C(X)$ with $\dim X = 0$, there exists a λ-graph system $\mathfrak{L}$ such that the subshift Λ_ρ is the subshift $\Lambda_\mathfrak{L}$ presented by $\mathfrak{L}$ and the C^*-algebra $\mathcal{O}_\rho$ is the C^*-algebra $\mathcal{O}_\mathfrak{L}$ associated with $\mathfrak{L}$. If in particular, $\mathcal{A} = \mathbb{C}^N$, the subshift Λ_ρ is a sofic shift and $\mathcal{O}_\rho$ is a Cuntz-Krieger algebra. If $\Sigma = \{\alpha\}$ an automorphism α of a unital C^*-algebra $\mathcal{A}$, the C^*-algebra $\mathcal{O}_\rho$ is the ordinary crossed product $\mathcal{A} \times_\alpha \mathbb{Z}$.

G. Robertson-T. Steger [38] have initiated a certain study of higher dimensional analogue of Cuntz-Krieger algebras from the view point of tiling systems of 2-dimensional plane. After their work, A. Kumjian-D. Pask [16] have generalized

their construction to introduce the notion of higher rank graphs and its C^*-algebras. The C^*-algebras constructed from higher rank graphs are called the higher rank graph C^*-algebras. Since then, there have been many studies on these C^*-algebras by many authors (see for example [8], [10], [16], [36], [31], [38], etc.).

M. Nasu in [29] has introduced the notion of textile system which is useful in analyzing automorphisms and endomorphisms of topological Markov shifts. A textile system also gives rise to a two-dimensional tiling called Wang tiling. Among textile systems, LR textile systems have specific properties that consist of two commuting symbolic matrices $\mathcal{MP} \cong \mathcal{PM}$. In [25], the author has extended the notion of textile systems to λ-graph systems and has defined a notion of textile systems on λ-graph systems, which are called textile λ-graph systems for short. C^*-algebras associated to textile systems have been initiated by V. Deaconu ([8]).

In this paper, we will extend the notion of C^*-symbolic dynamical system to C^*-textile dynamical system which is a higher dimensional analogue of C^*-symbolic dynamical system. The C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ consists of two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with the following commutation relations between ρ and η through κ. Set

$$\Sigma_{\rho\eta} = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_\alpha \neq 0\}, \quad \Sigma_{\eta\rho} = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_\beta \circ \eta_a \neq 0\}.$$

We require that there exists a bijection $\kappa : \Sigma_{\rho\eta} \longrightarrow \Sigma_{\eta\rho}$, which we fix and call a specification. Then the required commutation relations are

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{1.1}$$

C^*-textile dynamical systems provide two-dimensional tilings and C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$. The C^*-algbebra $\mathcal{O}^\kappa_{\rho,\eta}$ is defined to be the universal C^*-algebra $C^*(x, S_\alpha, T_a; x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta)$ generated by $x \in \mathcal{A}$ and two family of partial isometries $S_\alpha, \alpha \in \Sigma^\rho$, $T_a, a \in \Sigma^\eta$ subject to the following relations called $(\rho, \eta; \kappa)$:

$$\sum_{\beta \in \Sigma^\rho} S_\beta S_\beta^* = 1, \qquad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x), \tag{1.2}$$

$$\sum_{b \in \Sigma^\eta} T_b T_b^* = 1, \qquad x T_a T_a^* = T_a T_a^* x, \qquad T_a^* x T_a = \eta_a(x), \tag{1.3}$$

$$S_\alpha T_b = T_a S_\beta \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta) \tag{1.4}$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$.

We will construct a tiling system in the plane from a C^*-textle dynamical system. The resulting tiling system is a two-dimensional subshift.

In this paper, we will study the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$. We will introduce a condition called (I) on $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which will be studied as a generalization of the condition (I) on C^*-symbolic dynamical system [23](cf. [22]) (and hence on a finite matrix of Cuntz-Krieger [7]). Under the assumption that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I), the simplicity conditions of the algebra $\mathcal{O}^\kappa_{\rho,\eta}$ will be discussed in Section 4. We will show the following

Theorem 1.1. *Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-textile dynamical system satisfying condition (I). Then the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ is the unique C^*-algebra subject to the relations $(\rho, \eta; \kappa)$. If $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is irreducible, $\mathcal{O}^\kappa_{\rho,\eta}$ is simple.*

We denote by $Z_\mathcal{A}$ the center of $\mathcal{A}$. We next assume that $\rho_\alpha(Z_\mathcal{A}) \subset Z_\mathcal{A}, \alpha \in \Sigma^\rho$ and $\eta_a(Z_\mathcal{A}) \subset Z_\mathcal{A}, a \in \Sigma^\eta$. All examples of C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ appearing in the previous papers [24], [27] satisfy the conditions $\rho_\alpha(Z_\mathcal{A}) \subset$

$Z_{\mathcal{A}}, \alpha \in \Sigma$. A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to form squares if the C^*-subalgebra of $\mathcal{A}$ generated by projections $\rho_\alpha(1), \alpha \in \Sigma^\rho$ and the C^*-subalgebra of $\mathcal{A}$ generated by projections $\eta_a(1), a \in \Sigma^\eta$ coincide. We will restrict our interest to the C^*-textile dynamical systems forming squares. Then the K-theory formulae hold as in the following way:

Theorem 1.2. *Suppose that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ froms squares. There exists short exact sequences for $K_0(\mathcal{O}^\kappa_{\rho,\eta})$ and $K_1(\mathcal{O}^\kappa_{\rho,\eta})$ such as*

$$\begin{aligned} 0 &\longrightarrow K_0(\mathcal{A})/(1-\lambda_\eta)K_0(\mathcal{A}) + (1-\lambda_\rho)K_0(\mathcal{A}) \\ &\longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta}) \\ &\longrightarrow \operatorname{Ker}(1-\lambda_\eta) \cap \operatorname{Ker}(1-\lambda_\rho) \ \textit{in} \ K_0(\mathcal{A}) \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\longrightarrow \operatorname{Ker}(1-\lambda_\eta) \ \textit{in} \ K_0(\mathcal{A})/(1-\lambda_\rho)(\operatorname{Ker}(1-\lambda_\eta) \ \textit{in} \ K_0(\mathcal{A})) \\ &\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta}) \\ &\longrightarrow \operatorname{Ker}(1-\lambda_\rho) \ \textit{in} \ K_0(\mathcal{A})/(1-\lambda_\eta)K_0(\mathcal{A}) \longrightarrow 0 \end{aligned}$$

where the endomorphisms $\lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ are defined by

$$\begin{aligned} \lambda_\rho([p]) &= \sum_{\alpha \in \Sigma^\rho} [\rho_\alpha(p)] \in K_0(\mathcal{A}) \ \textit{for} \ [p] \in K_0(\mathcal{A}), \\ \lambda_\eta([p]) &= \sum_{a \in \Sigma^\eta} [\eta_a(p)] \in K_0(\mathcal{A}) \ \textit{for} \ [p] \in K_0(\mathcal{A}). \end{aligned}$$

Let A, B be $N \times N$ matrices with entries in nonnegative integers such that

$$AB = BA.$$

Let G_A, G_B be labeled directed graphs whose transition matrices are A, B respectively. Let $\mathcal{M}_A, \mathcal{M}_B$ denote symbolic marices for G_A, G_B whose conponents consist of formal sums of directed edges respectively. By the condition $AB = BA$, one may take a bijection $\kappa : \Sigma^{AB} \longrightarrow \Sigma^{BA}$ which gives rise to a specified equivalence $\mathcal{M}_A\mathcal{M}_B \overset{\kappa}{\cong} \mathcal{M}_A\mathcal{M}_B$. We then have a C^*-textile dynamical system written as

$$(\mathcal{A}, \rho^A, \rho^B, \Sigma^A, \Sigma^B, \kappa).$$

The associated C^*-algebra is denoted by $\mathcal{O}^\kappa_{A,B}$. The C^*-algebra $\mathcal{O}^\kappa_{A,B}$ is realized as a 2-graph C^*-algebra constructed from Kumjian and Pask. It is also seen in Deaconu's paper in [8].

Proposition 1.3. *Keep the above situations. There exist short exact sequences:*

(i)

$$\begin{aligned} 0 &\longrightarrow \mathbb{Z}^N/((1-A)\mathbb{Z}^N + (1-B)\mathbb{Z}^N) \\ &\longrightarrow K_0(\mathcal{O}^\kappa_{A,B}) \\ &\longrightarrow \operatorname{Ker}(1-A) \cap \operatorname{Ker}(1-B) \ \textit{in} \ \mathbb{Z}^N \longrightarrow 0 \end{aligned}$$

(ii)

$$\begin{aligned} 0 &\longrightarrow \operatorname{Ker}(1-B) \ \textit{in} \ \mathbb{Z}^N/(1-A)(\operatorname{Ker}(1-B) \ \textit{in} \ \mathbb{Z}^N) \\ &\longrightarrow K_1(\mathcal{O}^\kappa_{A,B}) \\ &\longrightarrow \operatorname{Ker}(1-\bar{A}) \ \textit{in} \ \mathbb{Z}^N/(1-B)\mathbb{Z}^N \longrightarrow 0, \end{aligned}$$

where $\bar{A}$ is an endomorphism on the abelian group $\mathbb{Z}^N/(1-B)\mathbb{Z}^N$ induced by the matrix A.

Throughout the paper, we will denote by $\mathbb{Z}_+$ and by $\mathbb{N}$ the sets of nonnegative integers and the set of positive integers respectively.

2. λ-graph systems, C^*-symbolic dynamical systems and their C^*-algebras

In this section, we will briefly review λ-graph systems and C^*-symbolic dynamical systems. Throughout the section, Σ denotes a finite set with its discrete topology, that is called an alphabet. Each element of Σ is called a symbol. Let $\Sigma^{\mathbb{Z}}$ be the infinite product space $\prod_{i\in\mathbb{Z}}\Sigma_i$, where $\Sigma_i=\Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}$ given by $\sigma((x_i)_{i\in\mathbb{Z}})=(x_{i+1})_{i\in\mathbb{Z}}$ is called the full shift over Σ. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda)=\Lambda$. The topological dynamical system $(\Lambda,\sigma|_\Lambda)$ is called a two-sided subshift, written as Λ for brevity.

There is a class of subshifts called sofic shifts, that are presented by finite labeled graphs. λ-graph systems are generalization of finite labeled graphs. Any subshift is presented by a λ-graph system. Let $\mathfrak{L}=(V,E,\lambda,\iota)$ be a λ-graph system over Σ with vertex set $V=\cup_{l\in\mathbb{Z}_+}V_l$ and edge set $E=\cup_{l\in\mathbb{Z}_+}E_{l,l+1}$ that is labeled with symbols in Σ by a map $\lambda: E\to\Sigma$, and that is supplied with surjective maps $\iota(=\iota_{l,l+1}): V_{l+1}\to V_l$ for $l\in\mathbb{Z}_+$. Here the vertex sets $V_l, l\in\mathbb{Z}_+$ are finite disjoint sets. Also $E_{l,l+1}, l\in\mathbb{Z}_+$ are finite disjoint sets. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1} respectively. Every vertex in V has a successor and every vertex in V_l for $l\in\mathbb{N}$ has a predecessor. It is then required that for vertices $u\in V_{l-1}$ and $v\in V_{l+1}$, there exists a bijective correspondence between the set of edges $e\in E_{l,l+1}$ such that $t(e)=v,\iota(s(e))=u$ and the set of edges $f\in E_{l-1,l}$ such that $s(f)=u, t(f)=\iota(v)$, preserving thier labels ([**?**]). We henceforth assume that $\mathfrak{L}$ is left-resolving, which means that $t(e)\neq t(f)$ whenever $\lambda(e)=\lambda(f)$ for $e,f\in E_{l,l+1}$. Let us denote by $\{v_1^l,\dots,v_{m(l)}^l\}$ the vertex set V_l at level l. For $i=1,2,\dots,m(l),\ j=1,2,\dots,m(l+1),\ \alpha\in\Sigma$ we put

$$A_{l,l+1}(i,\alpha,j)=\begin{cases}1 & \text{if } s(e)=v_i^l,\lambda(e)=\alpha,t(e)=v_j^{l+1} \text{ for some } e\in E_{l,l+1},\\ 0 & \text{otherwise,}\end{cases}$$

$$I_{l,l+1}(i,j)=\begin{cases}1 & \text{if } \iota_{l,l+1}(v_j^{l+1})=v_i^l,\\ 0 & \text{otherwise.}\end{cases}$$

The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with $\mathfrak{L}$ is the universal C^*-algebra generated by partial isometries $S_\alpha,\alpha\in\Sigma$ and projections $E_i^l, i=1,2,\dots,m(l),\ l\in\mathbb{Z}_+$ subject

to the following operator relations called ($\mathfrak{L}$):

$$\sum_{\beta\in\Sigma} S_\beta S_\beta^* = 1, \tag{2.1}$$

$$\sum_{i=1}^{m(l)} E_i^l = 1, \qquad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j)E_j^{l+1}, \tag{2.2}$$

$$S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*, \tag{2.3}$$

$$S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j)E_j^{l+1}, \tag{2.4}$$

for $i = 1,2,\dots,m(l), l \in \mathbb{Z}_+, \alpha \in \Sigma$. If $\mathfrak{L}$ satisfies λ-condition (I) and is λ-irreducible, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite ([23], [22]).

Let $\mathcal{A}_{\mathfrak{L},l}$ be the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the projections $E_i^l, i = 1,\dots,m(l)$. We denote by $\mathcal{A}_{\mathfrak{L}}$ the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the all projections $E_i^l, i = 1,\dots,m(l), l \in \mathbb{Z}_+$. We denote by $\iota : \mathcal{A}_{\mathfrak{L},l} \hookrightarrow \mathcal{A}_{\mathfrak{L},l+1}$ the natural inclusion. Hence the algebra $\mathcal{A}_{\mathfrak{L}}$ is the inductive limit $\varinjlim_{\iota} \mathcal{A}_{\mathfrak{L},l}$ of the inclusions. For $\alpha \in \Sigma$, put

$$\rho_\alpha^{\mathfrak{L}}(X) = S_\alpha^* X S_\alpha \qquad \text{for} \qquad X \in \mathcal{A}_{\mathfrak{L}}.$$

Then $\{\rho_\alpha^{\mathfrak{L}}\}_{\alpha\in\Sigma}$ yields a family of $*$-endomorphisms of $\mathcal{A}_{\mathfrak{L}}$ such that $\rho_\alpha^{\mathfrak{L}}(1) \neq 0$, $\sum_{\alpha\in\Sigma}\rho_\alpha^{\mathfrak{L}}(1) \geq 1$ and for any nonzero $x \in \mathcal{A}_{\mathfrak{L}}$, $\rho_\alpha^{\mathfrak{L}}(x) \neq 0$ for some $\alpha \in \Sigma$.

The situations above are generalized to C^*-symbolic dynamical systems as follows.

Let $\mathcal{A}$ be a unital C^*-algebra. In what follows, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit $1_{\mathcal{A}}$ of $\mathcal{A}$. The unit $1_{\mathcal{A}}$ is denoted by 1 unless we specify. For an alphabet Σ, a finite family of endomorphisms $\rho_\alpha, \alpha \in \Sigma$ of $\mathcal{A}$ is said to be *essential* if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with $\mathcal{A}$. It is said to be *faithful* if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$.

Definition ([24]). A C^*-*symbolic dynamical system* is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital C^*-algebra $\mathcal{A}$ and an essential and faithful finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of $\mathcal{A}$. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift Λ_ρ over Σ such that a word $\alpha_1 \cdots \alpha_k$ of Σ is admissible for Λ_ρ if and only if $(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$ ([24, Proposition 2.1]). Denote by $B_k(\Lambda_\rho)$ the set of admissible words of Λ_ρ respectively with length k. Put $B_*(\Lambda_\rho) = \cup_{k=0}^{\infty} B_k(\Lambda_\rho)$, where $B_0(\Lambda_\rho), B_0(\Lambda_\eta)$ denote the empty word. We say that *a subshift* Λ *acts on a* C^*-*algebra* $\mathcal{A}$ if there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift Λ_ρ is Λ. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *central* if $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$ for all $\alpha \in \Sigma$. In this case, essentiality of the endomorphisms $\rho_\alpha, \alpha \in \Sigma$ is equivalent to the condition that $\rho_\alpha(1) \neq 0, \alpha \in \Sigma$ and the inequality $\sum_{\alpha\in\Sigma}\rho_\alpha(1) \geq 1$ holds. All of the examples appeared in the papers [24], [27] are central in this sense. We will henceforth assume that C^*-symbolic dynamical systems are all central.

As in the above discussion we have a C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ from a λ-graph system $\mathfrak{L}$ such that the C^*-algebra $\mathcal{A}_{\mathfrak{L}}$ is $C(\Omega_{\mathfrak{L}})$ with $\dim\Omega_{\mathfrak{L}} = 0$, and the subshift $\Lambda_{\rho^{\mathfrak{L}}}$ coincides with the subshift $\Lambda_{\mathfrak{L}}$ presented by $\mathfrak{L}$. Conversely, for a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, if the algebra $\mathcal{A}$ is $C(X)$ with $\dim X =$

0, there exists a λ-graph system $\mathfrak{L}$ over Σ such that the associated C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}, \rho, \Sigma)$ ([24, Theorem 2.4]).

The C^*-algebra $\mathcal{O}_\rho$ associated with a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ has been originally constructed in [24] as a C^*-algebra by using the Pimsner's general construction of C^*-algebras from Hilbert C^*-bimodules [35] (cf. [13] etc.). It is called *the C^*-symbolic crossed product of $\mathcal{A}$ by the subshift Λ_ρ*, and realized as the universal C^*-algebra $C^*(x, S_\alpha; x \in \mathcal{A}, \alpha \in \Sigma)$ generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma$ subject to the following relations called (ρ):

$$\sum_{\beta\in\Sigma} S_\beta S_\beta^* = 1, \qquad xS_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. Furthermore for $\alpha_1, \dots, \alpha_k \in \Sigma$, a word $(\alpha_1, \dots, \alpha_k)$ is admissible for the subshift Λ_ρ if and only if $S_{\alpha_1} \cdots S_{\alpha_k} \neq 0$ ([24, Proposition 3.1]). The C^*-algebra $\mathcal{O}_\rho$ is a generalization of the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$.

Let α be an automorphism of a unital C^*-algebra $\mathcal{A}$. Put $\Sigma = \{\alpha\}$ and $\rho_\alpha = \alpha$. The C^*-algebra $\mathcal{O}_\rho$ for the C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is the ordinary crossed product $\mathcal{A} \times_\alpha \mathbb{Z}$.

3. C^*-textile dynamical systems and their C^*-algebras

Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-textile dynamical system. It consists of two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with the following commutation relations through κ. Set

$$\Sigma_{\rho\eta} = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_\alpha \neq 0\}, \quad \Sigma_{\eta\rho} = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_\beta \circ \eta_a \neq 0\}.$$

Let $\kappa : \Sigma_{\rho\eta} \longrightarrow \Sigma_{\eta\rho}$ be a bijection, which is called a specification. Then the required commutation relations are

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{3.1}$$

C^*-textile dynamical systems will yield a two-dimensional subshift $X_{\rho,\eta}^\kappa$ and a C^*-algebra $\mathcal{O}_{\rho,\eta}^\kappa$.

Let Σ be a finite set. The two-dimensional full shift over Σ is defined to be

$$\Sigma^{\mathbb{Z}^2} = \{(x_{i,j})_{(i,j)\in\mathbb{Z}^2} \mid x_{i,j} \in \Sigma\}.$$

An element $x \in \Sigma^{\mathbb{Z}^2}$ is regarded as a function $x : \mathbb{Z}^2 \longrightarrow \Sigma$ which is called a configuration on $\mathbb{Z}^2$. For $x \in \Sigma^{\mathbb{Z}^2}$ and $F \subset \mathbb{Z}^2$, let x_F denote the restriction of x to F. For a vector $m = (m_1, m_2) \in \mathbb{Z}^2$, let $\sigma^m : \Sigma^{\mathbb{Z}^2} \longrightarrow \Sigma^{\mathbb{Z}^2}$ be the translation along vector m defined by

$$\sigma^m((x_{i,j})_{(i,j)\in\mathbb{Z}^2}) = (x_{i+m_1,j+m_2})_{(i,j)\in\mathbb{Z}^2}.$$

A subset $X \subset \Sigma^{\mathbb{Z}^2}$ is said to be translation invariant if $\sigma^m(X) = X$ for all $m \in \mathbb{Z}^2$. It is obvious to see that a subset $X \subset \Sigma^{\mathbb{Z}^2}$ is translation invariant if ond only if X is invariant only both horizontaly and vertically, that is, $\sigma^{(1,0)}(X) = X$ and $\sigma^{(0,1)}(X) = X$. For $k \in \mathbb{Z}$, put

$$[-k, k]^2 = \{(i, j) \in \mathbb{Z}^2 \mid -k \leq i, j \leq k\} = [-k, k] \times [-k, k].$$

A metric d on $\Sigma^{\mathbb{Z}^2}$ is defined by for $x, y \in \Sigma^{\mathbb{Z}^2}$ with $x \neq y$

$$d(x,y) = \frac{1}{2^k} \quad \text{if} \quad x_{(0,0)} = y_{(0,0)},$$

where $k = \max\{k \in \mathbb{Z}_+ \mid x_{[-k,k]^2} = y_{[-k,k]^2}\}$. If $x_{(0,0)} \neq y_{(0,0)}$, put $k = -1$ on the above definition. If $x = y$, we set $d(x,y) = 0$. A two-dimensional subshift X is a closed, translation invariant subset of $\Sigma^{\mathbb{Z}^2}$ (cf. [18, p.467]). There is an equivalent definition of two dimensional subshift based on lists of forbidden patterns as follows: A shape is a finite subset $F \subset \mathbb{Z}^2$. A pattern f on a shape F is a function $f : F \longrightarrow \Sigma$. For a list $\mathfrak{F}$ of patterns, put

$$X_{\mathfrak{F}} = \{(x_{i,j})_{(i,j)\in\mathbb{Z}^2} \mid \sigma^m(x)_F \notin \mathfrak{F} \text{ for all } m \in \mathbb{Z}^2 \text{ and } F \subset \mathbb{Z}^2\}.$$

It is well-known that a subset $X \subset \Sigma^{\mathbb{Z}^2}$ is a two-dimensional subshift if and only if there exists a list of patterns $\mathfrak{F}$ such that $X = X_{\mathfrak{F}}$.

We will define a certain property of two-dimensional subshift as follows:

Definition. A two-dimensional subshift X is said to have *diagonal property* if for $(x_{i,j})_{(i,j)\in\mathbb{Z}^2}, (y_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X$, the conditions $x_{i,j} = y_{i,j}, x_{i+1,j-1} = y_{i+1,j-1}$ imply $x_{i,j-1} = y_{i,j-1}, x_{i+1,j} = y_{i+1,j}$. A two-dimensional subshift having diagonal property is called *textile dynamical system.*

Lemma 3.1. *If a two dimensional subshift X has diagonal propety, then for $x \in X$ and $(i,j) \in \mathbb{Z}^2$, the configuration x is determined by the diagonal line $(x_{i+n,j-n})_{n\in\mathbb{Z}}$ through (i,j).*

Proof. By the diagonal property, the sequence $(x_{i+n,j-n})_{n\in\mathbb{Z}}$ determines both the sequences $(x_{i+1+n,j-n})_{n\in\mathbb{Z}}$ and $(x_{i-1+n,j-n})_{n\in\mathbb{Z}}$. Repeating this way, one sees that the sequence $(x_{i+n,j-n})_{n\in\mathbb{Z}}$ determines $x_{n,m}$ for all $(n,m) \in \mathbb{Z}^2$. $\square$

Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-textile dynamical system. We set

$$\Sigma_\kappa = \{\omega = (\alpha, b, a, \beta) \in \Sigma^\rho \times \Sigma^\eta \times \Sigma^\eta \times \Sigma^\rho \mid \kappa(\alpha, b) = (a, \beta)\}$$

For $\omega = (\alpha, b, a, \beta)$, since $\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a$ as endomorphism on $\mathcal{A}$, one may identify the quadruplet (α, b, a, β) with the endomorphism $\eta_b \circ \rho_\alpha (= \rho_\beta \circ \eta_a)$ on $\mathcal{A}$ which we will denote by simply ω. Define maps $t, b : \Sigma_\kappa \longrightarrow \Sigma^\rho$ and $l, r : \Sigma_\kappa \longrightarrow \Sigma^\rho$ by setting

$$t(\omega) = \alpha, \quad b(\omega) = \beta, \quad l(\omega) = a, \quad r(\omega) = b$$

$$\begin{array}{ccc} \cdot & \xrightarrow{\alpha = t(\omega)} & \cdot \\ {\scriptstyle a = l(\omega)}\Big\downarrow & & \Big\downarrow{\scriptstyle b = r(\omega)} \\ \cdot & \xrightarrow[\beta = b(\omega)]{} & \cdot \end{array}$$

A configuration $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_\kappa^{\mathbb{Z}^2}$ is said to be *paived* if the following conditions hold

$$t(\omega_{i,j}) = b(\omega_{i,j+1}), \quad r(\omega_{i,j}) = l(\omega_{i+1,j}), \quad l(\omega_{i,j}) = r(\omega_{i-1,j}), \quad b(\omega_{i,j}) = t(\omega_{i,j-1})$$

for all $(i,j) \in \mathbb{Z}^2$.

For a textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, we set

$$X_{\rho,\eta}^\kappa = \{(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_\kappa^{\mathbb{Z}^2} \mid (\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \text{ is paved and }$$
$$\omega_{i+n,j-n} \circ \omega_{i+n-1,j-n+1} \circ \cdots \circ \omega_{i+1,j-1} \circ \omega_{i,j} \neq 0 \text{ for all } (i,j) \in \mathbb{Z}^2, n \in \mathbb{N}\}$$

Lemma 3.2. *Suppose that $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2}$ is paved. Then $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^{\kappa}_{\rho,\eta}$ if and only if*

$$\rho_{b(\omega_{i+n,j-m})} \circ \cdots \circ \rho_{b(\omega_{i+1,j-m})} \circ \rho_{b(\omega_{i,j-m})} \circ \eta_{l(\omega_{i,j-m})} \circ \cdots \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})} \neq 0$$

for all $(i,j) \in \mathbb{Z}^2$, $n, m \in \mathbb{Z}_+$.

$$\begin{tikzcd}
\cdot \arrow[d, "l(\omega_{i,j})"'] & & & & \\
\cdot \arrow[d, "l(\omega_{i,j-1})"'] & & & & \\
\cdot & & & & \\
\vdots & & & & \\
\cdot \arrow[d, "l(\omega_{i,j-m})"'] & & & & \\
\cdot \arrow[r, "b(\omega_{i,j-m})"'] & \cdot \arrow[r, "b(\omega_{i+1,j-m})"'] & \cdots \arrow[r, "b(\omega_{i+n,j-m})"'] & \cdot
\end{tikzcd}$$

Proof. Suppose that $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^{\kappa}_{\rho,\eta}$. For $(i,j) \in \mathbb{Z}^2$, $n, m \in \mathbb{Z}_+$, we may assume that $m \geq n$. Since

$$\begin{aligned}
0 \neq & \omega_{i+m,j-m} \circ \cdots \circ \omega_{i+n+1,j-m} \circ \omega_{i+n,j-m} \circ \cdots \circ \omega_{i,j-m} \circ \cdots \circ \omega_{i+1,j-1} \circ \omega_{i,j} \\
= & \omega_{i+m,j-m} \circ \cdots \circ \omega_{i+n+1,j-m} \circ \rho_{b(\omega_{i+n,j-m})} \circ \cdots \circ \rho_{b(\omega_{i+1,j-m})} \circ \rho_{b(\omega_{i,j-m})} \\
& \circ \eta_{l(\omega_{i,j-m})} \cdots \circ \eta_{l(\omega_{i,j-m})} \circ \cdots \circ \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})},
\end{aligned}$$

one has

$$\rho_{b(\omega_{i+n,j-m})} \circ \cdots \circ \rho_{b(\omega_{i+1,j-m})} \circ \rho_{b(\omega_{i,j-m})} \circ \eta_{l(\omega_{i,j-m})} \circ \cdots \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})} \neq 0.$$

Converse implications is clear by the equality:

$$\begin{aligned}
& \omega_{i+n,j-n} \circ \cdots \circ \omega_{i,j-n} \circ \cdots \circ \omega_{i,j-1} \circ \omega_{i,j} \\
& = \rho_{b(\omega_{i+n,j-n})} \circ \cdots \circ \rho_{b(\omega_{i,j-n})} \circ \eta_{l(\omega_{i,j-n})} \cdots \circ \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})}.
\end{aligned}$$

□

Proposition 3.3. *For $(\mathcal{A}, \rho, \eta, \Sigma^{\rho}, \Sigma^{\eta}, \kappa)$, $X^{\kappa}_{\rho,\eta}$ is a two-dimensional subshift having diagonal property, that is, $X^{\kappa}_{\rho,\eta}$ is a textile dynamical system.*

Proof. It is easy to see that the set

$$E = \{(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_{\kappa}^{\mathbb{Z}^2} \mid (\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \text{ is paved } \}$$

is closed, because its complement is open. The following set

$$\begin{aligned}
U = \{(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_{\kappa}^{\mathbb{Z}^2} \mid & \omega_{k+n,l-n} \circ \omega_{k+n-1,l-n+1} \circ \cdots \circ \omega_{k+1,l-1} \circ \omega_{k,l} = 0 \\
& \text{for some } (k,l) \in \mathbb{Z}^2, n \in \mathbb{N}\}
\end{aligned}$$

is open in $\Sigma_{\kappa}^{\mathbb{Z}^2}$. As the equality $X^{\kappa}_{\rho,\eta} = E \cap U^c$ holds, the set $X^{\kappa}_{\rho,\eta}$ is closed. It is also obvious that $X^{\kappa}_{\rho,\eta}$ is translation invariant so that $X^{\kappa}_{\rho,\eta}$ is a textile dynamical system. It is easy to see that $X^{\kappa}_{\rho,\eta}$ has diagonal property. □

We call $X^\kappa_{\rho,\eta}$ the textile dynamical system associated with $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$.

Let us now define a subshift X_{δ^κ} over Σ_κ, which consists of diagonal sequences of $X^\kappa_{\rho,\eta}$ as follows:

$$X_{\delta^\kappa} = \{(\omega_{n,-n})_{n\in\mathbb{Z}} \in \Sigma_\kappa^{\mathbb{Z}} \mid (\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^\kappa_{\rho,\eta}\}.$$

By Lemma 2.1, an element $(\omega_{n,-n})_{n\in\mathbb{Z}}$ of X_{δ^κ} may be extend to $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^\kappa_{\rho,\eta}$ in a unique way. Hence the one-dimensional subshift X_{δ^κ} determines the two-dimensional subshift $X^\kappa_{\rho,\eta}$. Therefore we have

Lemma 3.4. *For $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$, the two-dimensional subshift $X^\kappa_{\rho,\eta}$ is not empty if and only if the one-dimensional subshift X_{δ^κ} is not empty.*

For $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$, we will have a C^*-symbolic dynamical system $(\mathcal{A},\delta^\kappa,\Sigma_\kappa)$ in Section 5. It presents the subshift X_{δ^κ}. Since a subshift presented by a C^*-symbolic dynamical system is always not empty, one sees that $X^\kappa_{\rho,\eta}$ is not empty.

Proposition 3.5. *For $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$, the two-dimensional subshift $X^\kappa_{\rho,\eta}$ is not empty.*

The C^*-algbebra $\mathcal{O}^\kappa_{\rho,\eta}$ is defined to be the universal C^*-algebra $C^*(x,S_\alpha,T_a; x\in \mathcal{A},\alpha\in\Sigma^\rho, a\in\Sigma^\eta)$ generated by $x\in\mathcal{A}$ and partial isometries $S_\alpha,\alpha\in\Sigma^\rho$, $T_a, a\in\Sigma^\eta$ subject to the following relations called (ρ,η):

$$\sum_{\beta\in\Sigma^\rho} S_\beta S_\beta^* = 1, \qquad xS_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x), \tag{3.2}$$

$$\sum_{b\in\Sigma^\eta} T_b T_b^* = 1, \qquad xT_a T_a^* = T_a T_a^* x, \qquad T_a^* x T_a = \eta_a(x), \tag{3.3}$$

$$S_\alpha T_b = T_a S_\beta \qquad \text{if} \quad \kappa(\alpha,b) = (a,\beta) \tag{3.4}$$

for all $x\in\mathcal{A}$ and $\alpha\in\Sigma^\rho, a\in\Sigma^\eta$. We will study the algebra $\mathcal{O}^\kappa_{\rho,\eta}$. If $\kappa(\alpha,b)=(a,\beta)$, we write as $(\alpha,b)\overset{\kappa}{\cong}(a,\beta)$.

Lemma 3.6. *For $\alpha\in\Sigma^\rho, a\in\Sigma^\eta$, one has $T_a^*S_\alpha \neq 0$ if and only if there exist $b\in\Sigma^\eta,\beta\in\Sigma^\rho$ such that $(\alpha,b)\overset{\kappa}{\cong}(a,\beta)$.*

Proof. Suppose that $T_a^*S_\alpha\neq 0$. As $T_a^*S_\alpha = \sum_{b'\in\Sigma^\eta} T_a^* S_\alpha T_{b'}T_{b'}^*$, there exists $b'\in \Sigma^\eta$ such that $T_a^*S_\alpha T_{b'}\neq 0$. Hence $\eta_{b'}\circ\rho_\alpha\neq 0$ so that $(\alpha,b')\in\Sigma^{\rho\eta}$. Then one may find $(a',\beta')\in\Sigma^\rho$ such that $(\alpha,b')\overset{\kappa}{\cong}(a',\beta')$ and hence $S_\alpha T_{b'} = T_{a'}S_{\beta'}$. Since $0\neq T_a^*S_\alpha T_{b'} = T_a^*T_{a'}S_{\beta'}$, one sees that $a=a'$. Putting $b=b',\beta=\beta'$, we have $\kappa(\alpha,b)=(a,\beta)$.

Suppose next that $\kappa(\alpha,b)=(a,\beta)$. Since $\eta_b\circ\rho_\alpha = \rho_\beta\circ\eta_a\neq 0$, one has $0\neq S_\alpha T_b = T_a S_\beta$. It follows that $S_\beta^*T_a^*S_\alpha T_b = (T_aS_\beta)^*T_aS_\beta$ so that $T_a^*S_\alpha\neq 0$. ☐

Lemma 3.7. *For $\alpha\in\Sigma^\rho, a\in\Sigma^\eta$, we have*

$$T_a^*S_\alpha = \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\beta\eta_b(\rho_\alpha(1))T_b^* \tag{3.5}$$

and hence

$$S_\alpha^*T_a = \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} T_b\rho_\beta(\eta_a(1))S_\beta^*. \tag{3.6}$$

Proof. We may assume that $T_a^* S_\alpha \neq 0$. One has $T_a^* S_\alpha = \sum_{b' \in \Sigma^\eta} T_a^* S_\alpha T_{b'} T_{b'}^*$. For $b' \in \Sigma^\eta$ with $(\alpha, b') \in \Sigma^{\rho,\eta}$, and for $\beta' \in \Sigma^\rho$ such that $\kappa(\alpha, b') = (a', \beta')$ for some $a' \in \Sigma^\eta$, one has

$$T_a^* S_\alpha T_{b'} T_{b'}^* = T_a^* T_{a'} S_{\beta'} T_{b'}^*.$$

Hence $T_a^* S_\alpha T_{b'} T_{b'}^* \neq 0$ implies $a = a'$. Since $T_a^* T_a = \eta_a(1)$ which commutes with $S_{\beta'} S_{\beta'}^*$, we have

$$T_a^* T_a S_{\beta'} T_{b'}^* = S_{\beta'} S_{\beta'}^* T_a^* T_a S_{\beta'} T_{b'}^* = S_{\beta'} \rho_{\beta'}(\eta_a(1)) T_{b'}^* = S_{\beta'} \eta_{b'}(\rho_\alpha(1)) T_{b'}^*.$$

It follows that

$$T_a^* S_\alpha = \sum_{\substack{b', \beta' \\ \kappa(\alpha, b') = (a, \beta')}} T_a^* T_a S_{\beta'} T_{b'}^* = \sum_{\substack{b', \beta' \\ \kappa(\alpha, b') = (a, \beta')}} S_{\beta'} \eta_{b'}(\rho_\alpha(1)) T_{b'}^*.$$

□

Hence we have

Lemma 3.8. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have*

$$T_a T_a^* S_\alpha S_\alpha^* = \sum_{\substack{b \\ \kappa(\alpha, b) = (a, \beta) \textit{for some } \beta}} S_\alpha T_b T_b^* S_\alpha^*. \tag{3.7}$$

Hence $T_a T_a^*$ commutes with $S_\alpha S_\alpha^*$.

Proof. By the preceding lemma, we have

$$\begin{aligned}
T_a T_a^* S_\alpha S_\alpha^* &= \sum_{\substack{b, \beta \\ \kappa(\alpha, b) = (a, \beta)}} T_a S_\beta \eta_b(\rho_\alpha(1)) T_b^* S_\alpha^* \\
&= \sum_{\substack{b, \beta \\ \kappa(\alpha, b) = (a, \beta)}} S_\alpha T_b \eta_b(\rho_\alpha(1)) T_b^* S_\alpha^* \\
&= \sum_{\substack{b, \beta \\ \kappa(\alpha, b) = (a, \beta)}} S_\alpha \rho_\alpha(1)) T_b T_b^* S_\alpha^* \\
&= \sum_{\substack{b \\ \kappa(\alpha, b) = (a, \beta) \text{for some } \beta}} S_\alpha T_b T_b^* S_\alpha^*.
\end{aligned}$$

□

More generally, we have

Lemma 3.9. *Suppose that $\mathcal{A}$ is commutative. For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, and $x, y \in \mathcal{A}$, we know that $T_a y T_a^*$ commutes with $S_\alpha x S_\alpha^*$.*

Proof. It follows that

$$
\begin{aligned}
T_a y T_a^* S_\alpha x S_\alpha^* &= T_a y \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\beta \eta_b(\rho_\alpha(1)) T_b^* x S_\alpha^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} T_a S_\beta S_\beta^* y S_\beta \eta_b(\rho_\alpha(1)) T_b^* x T_b T_b^* S_\alpha^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha T_b \rho_\beta(y) \eta_b(\rho_\alpha(1)) \eta_b(x) S_\beta^* T_a^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha T_b \eta_b(x) \eta_b(\rho_\alpha(1)) \rho_\beta(y) S_\beta^* T_a^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha x \rho_\alpha(1) T_b S_\beta^* y T_a^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha x S_\alpha^* S_\alpha T_b S_\beta^* T_a^* T_a y T_a^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha x (S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a) y T_a^*.
\end{aligned}
$$

Now if $(\alpha, b') \notin \Sigma^{\rho,\eta}$, then $S_\alpha T_{b'} = 0$. Hence

$$
\sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a = \sum_b S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a = S_\alpha^* T_a.
$$

Therefore we have

$$
T_a y T_a^* S_\alpha x S_\alpha^* = S_\alpha x S_\alpha^* T_a y T_a^*.
$$

$\square$

We set

$$
\begin{aligned}
\mathcal{D}_{\rho,\eta} =& C^*(S_\mu T_\zeta x T_\zeta^* S_\mu^* : \mu \in B_*(\Lambda_\rho), \zeta \in B_*(\Lambda_\eta), x \in \mathcal{A}), \\
\mathcal{D}_{j,k} =& C^*(S_\mu T_\zeta x T_\zeta^* S_\mu^* : \mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta), x \in \mathcal{A}) \text{ for } j,k \in \mathbb{Z}_+.
\end{aligned}
$$

By the commutation relation (3.5), one sees that

$$
\mathcal{D}_{j,k} = C^*(T_\xi S_\nu x S_\nu^* T_\xi^* : \nu \in B_j(\Lambda_\rho), \xi \in B_k(\Lambda_\eta), x \in \mathcal{A})
$$

The identities

$$
\begin{aligned}
S_\mu T_\zeta x T_\zeta^* S_\mu^* &= \sum_{a \in \Sigma^\eta} S_\mu T_{\zeta a} \eta_a(x) T_{\zeta a}^* S_\mu^*, \\
T_\xi S_\nu x S_\nu^* T_\xi^* &= \sum_{\alpha \in \Sigma^\rho} T_\xi S_{\nu\alpha} \rho_\alpha(x) S_{\nu\alpha}^* T_\xi^*
\end{aligned}
$$

for $x \in \mathcal{A}$ and $\mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta)$ yield the embeddings

$$
\mathcal{D}_{j,k} \hookrightarrow \mathcal{D}_{j,k+1}, \qquad \mathcal{D}_{j,k} \hookrightarrow \mathcal{D}_{j+1,k}
$$

respectively such that $\cup_{j,k \in \mathbb{Z}_+} \mathcal{D}_{j,k}$ is dense in $\mathcal{D}_{\rho,\eta}$.

Proposition 3.10. *If $\mathcal{A}$ is commutative, so is $\mathcal{D}_{\rho,\eta}$.*

Proof. The preceding lemma tells us that $\mathcal{D}_{1,1}$ is commutative. Suppose that the algebra $\mathcal{D}_{j,k}$ is commutative for a fixed $j, k \in \mathbb{N}$. We will show that the both algebras $\mathcal{D}_{j+1,k}$ and $\mathcal{D}_{j,k+1}$ are commutative. For the algebra $\mathcal{D}_{j+1,k}$, it consists of linear span of elements of the form:

$$S_\alpha x S_\alpha^* \qquad \text{for } x \in \mathcal{D}_{j,k}, \alpha \in \Sigma^\rho.$$

Let $x, y \in \mathcal{D}_{j,k}, \alpha, \beta \in \Sigma^\rho$. We will show that $S_\alpha x S_\alpha^*$ commutes with both $S_\beta y S_\beta^*$ and y. If $\alpha = \beta$, it is easy to see that $S_\alpha x S_\alpha^*$ commutes with $S_\alpha y S_\alpha^*$, because $\rho_\alpha(1) \in \mathcal{A} \subset \mathcal{D}_{j,k}$. If $\alpha \neq \beta$, both $S_\alpha x S_\alpha^* S_\beta y S_\beta^*$ and $S_\beta y S_\beta^* S_\alpha x S_\alpha^*$ are zeros. Since $S_\alpha^* y S_\alpha \in \mathcal{D}_{j-1,k} \subset \mathcal{D}_{j,k}$, one sees $S_\alpha^* y S_\alpha$ commutes with x. One also sees that $S_\alpha S_\alpha^* \in \mathcal{D}_{j,k}$ commutes with y. It follows that

$$S_\alpha x S_\alpha^* y = S_\alpha x S_\alpha^* y S_\alpha S_\alpha^* = S_\alpha S_\alpha^* y S_\alpha x S_\alpha^* = y S_\alpha x S_\alpha^*.$$

Hence the algebra $\mathcal{D}_{j+1,k}$ is commutative, and similarly so is $\mathcal{D}_{j,k+1}$. By induction, one knows that the algebras $\mathcal{D}_{j,k}$ are all commutative for all $j, k \in \mathbb{N}$. Since $\cup_{j,k\in\mathbb{N}} \mathcal{D}_{j,k}$ is dense in $\mathcal{D}_{\rho,\eta}$, $\mathcal{D}_{\rho,\eta}$ is commuatative. $\square$

Proposition 3.11. *Let $\mathcal{O}_{\rho,\eta}^{alg}$ be the dense $*$-subalgebra algebraically generated by elements $x \in \mathcal{A}$, $S_\alpha, \alpha \in \Sigma^\rho$ and $T_a, a \in \Sigma^\eta$. Then each element of $\mathcal{O}_{\rho,\eta}^{alg}$ is a finite linear combination of elements of the form:*

$$S_\mu T_\zeta x T_\xi^* S_\nu^* \qquad \text{for } x \in \mathcal{A}, \mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta) \tag{3.8}$$

where $S_\mu = S_{\mu_1} \cdots S_{\mu_k}, S_\nu = S_{\nu_1} \cdots S_{\nu_n}$ for $\mu = \mu_1 \cdots \mu_k, \nu = \nu_1 \cdots \nu_n$ and $T_\zeta = T_{\zeta_1} \cdots T_{\zeta_l}, T_\xi = T_{\xi_1} \cdots T_{\xi_m}$ for $\zeta = \zeta_1 \cdots \zeta_l, \xi = \xi_1 \cdots \xi_m$.

Proof. For $\alpha, \beta \in \Sigma^\rho$, $a, b \in \Sigma^\eta$ and $x \in \mathcal{A}$, we have

$$\begin{aligned} S_\alpha^* S_\beta &= \begin{cases} \rho_\alpha(1) \in \mathcal{A} & \text{if } \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases} \\ S_\alpha^* T_a &= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} T_b \rho_\beta(\eta_a(1)) S_\beta^*, \\ S_\alpha^* x &= \rho_\alpha(x) S_\alpha, \qquad S_\beta^* T_a^* = T_b^* S_\alpha^*. \end{aligned}$$

And also

$$\begin{aligned} T_a^* T_b &= \begin{cases} \eta_a(1) \in \mathcal{A} & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases} \\ T_a^* S_\alpha &= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\beta \eta_b(\rho_\alpha(1)) T_b^*, \\ T_a^* x &= \eta_a(x) T_a^*. \end{aligned}$$

Therefore we conclude that any element of $\mathcal{O}_{\rho,\eta}^{alg}$ is a finite linear combination of elements of the form: $S_\mu T_\zeta x T_\xi^* S_\nu^*$. $\square$

Similarly we have

Proposition 3.12. *Each element of $\mathcal{O}_{\rho,\eta}^{alg}$ is a finite linear combination of elements of the form:*

$$T_\zeta S_\mu x S_\nu^* T_\xi^* \qquad \text{for } x \in \mathcal{A}, \mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta). \tag{3.9}$$

In the rest of this section, we will have a C^*-symbolic dynamical system $(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ from $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, which presents the one-dimensional subshift X_{δ^κ} described in the previous section. For $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, define an endomorphism δ^κ_ω on $\mathcal{A}$ for $\omega \in \Sigma_\kappa$ by setting

$$\delta^\kappa_\omega(x) = \eta_b(\rho_\alpha(x))(= \rho_\beta(\eta_a(x))), \qquad x \in \mathcal{A}, \quad \omega = (\alpha, b, a, \beta) \in \Sigma_\kappa.$$

Lemma 3.13. *$(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ is a C^*-symbolic dynamical system that presents X_{δ^κ}.*

Proof. We will show that δ^κ is essential and faithful. Now both C^*-symbolic dynamical systems $(\mathcal{A}, \eta, \Sigma^\eta)$ and $(\mathcal{A}, \rho, \Sigma^\eta)$ are essential. We are further assuming that both C^*-symbolic dynamical systems $(\mathcal{A}, \eta, \Sigma^\eta)$ and $(\mathcal{A}, \rho, \Sigma^\eta)$ are central. Hence it is clear that $\delta^\kappa_\omega(Z_\mathcal{A}) \subset Z_\mathcal{A}$. By the inequalities

$$\sum_{\omega\in\Sigma_\kappa} \delta^\kappa_\omega(1) = \sum_{b\in\Sigma^\eta} \sum_{\alpha\in\Sigma^\rho} \eta_b(\rho_\alpha(1)) \ge \sum_{b\in\Sigma^\eta} \eta_b(1) \ge 1$$

$\{\delta^\kappa\}_{\omega\in\Sigma_\kappa}$ is essential. For any nonzero $x \in \mathcal{A}$, there exists $\alpha \in \Sigma^\rho$ such that $\rho_\alpha(x) \ne 0$ and there exists $b \in \Sigma^\eta$ such that $\eta_b(\rho_\alpha(x)) \ne 0$. This means that $\delta^\kappa_\omega(x) \ne 0$ for $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$. Hence δ^κ is faithful so that $(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ is a C^*-symbolic dynamical system. It is obvious that the presented subshift by $(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ is X_{δ^κ}. $\square$

Put

$$\widehat{X}^\kappa_{\rho,\eta} = \{(\omega_{i,-j})_{(i,j)\in\mathbb{N}^2} \in \Sigma_\kappa^{\mathbb{N}^2} \mid (\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^\kappa_{\rho,\eta}\}$$

and

$$\widehat{X}_{\delta^\kappa} = \{(\omega_{n,-n})_{n\in\mathbb{N}} \in \Sigma_\kappa^{\mathbb{N}} \mid (\omega_{i,j})_{(i,j)\in\mathbb{N}^2} \in \widehat{X}^\kappa_{\rho,\eta}\}.$$

The latter set $\widehat{X}_{\delta^\kappa}$ is the right one-sided subshift for X_{δ^κ}.

Lemma 3.14. *A configuration $(\omega_{i,-j})_{(i,j)\in\mathbb{N}^2} \in \widehat{X}^\kappa_{\rho,\eta}$ can extend to a whole configuration $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^\kappa_{\rho,\eta}$.*

Proof. For $(\omega_{i,-j})_{(i,j)\in\mathbb{N}^2} \in \widehat{X}^\kappa_{\rho,\eta}$, put $x_i = \omega_{i,-i}, i \in \mathbb{N}$ so that $x = (x_i)_{i\in\mathbb{N}} \in \widehat{X}_{\delta^\kappa}$. Since $\widehat{X}_{\delta^\kappa}$ is a one-sided subshift, there exists an extension $\tilde{x} \in X_{\delta^\kappa}$ to two-sided sequence such that $\tilde{x}_{[1,\infty)} = x$. By the diagonal property, $\tilde{x}$ determines a whole configuration $\tilde{\omega}$ to $\mathbb{Z}^2$ such that $\tilde{\omega} \in X^\kappa_{\delta,\eta}$ and $(\tilde{\omega}_{i,-i})_{i\in\mathbb{N}} = \tilde{x}$. Hence $\tilde{\omega}_{i,-j} = \omega_{i,-j}$ for all $i.j \in \mathbb{N}$. $\square$

Let $\mathfrak{D}_{\rho,\eta}$ be the C^*-subalgebra of $\mathcal{D}_{\rho,\eta}$ defined by

$$\begin{aligned}\mathfrak{D}_{\rho,\eta} &= C^*(S_\mu T_\zeta T^*_\zeta S^*_\mu : \mu \in B_*(\Lambda_\rho), \zeta \in B_*(\Lambda_\eta)\\ &= C^*(T_\xi S_\nu S^*_\nu T^*_\xi : \nu \in B_*(\Lambda_\rho), \xi \in B_*(\Lambda_\eta)\end{aligned}$$

which is a commutative C^*-subalgebra of $\mathcal{D}_{\rho,\eta}$. Put for $\mu = \mu_1 \cdots \mu_n \in B_*(\Lambda_\rho)$, $\zeta = \zeta_1 \cdots \zeta_m \in B_*(\Lambda_\eta)$ the cylinder set

$$U_{\mu,\zeta} = \{(\omega_{i,-j})_{(i,j)\in\mathbb{N}^2} \in \widehat{X}^\kappa_{\rho,\eta} \mid t(\omega_{i,-1}) = \mu_i, i = 1, \cdots, n, r(\omega_{n,-j}) = \zeta_j, j = 1, \cdots, m\}$$

The following lemma is direct.

Lemma 3.15. *$\mathfrak{D}_{\rho,\eta}$ is isomorphic to $C(\widehat{X}^\kappa_{\rho,\eta})$ through the corespondence such that $S_\mu T_\zeta T^*_\zeta S^*_\mu$ sends to $\chi_{U_{\mu,\zeta}}$, where $\chi_{U_{\mu,\zeta}}$ is the characteristic function for the cylinder set $U_{\mu,\zeta}$ on $\widehat{X}^\kappa_{\rho,\eta}$.*

4. Condition (I) for C^*-textile dynamical systems

The notion of condition (I) for finite square matrices with entries in $\{0,1\}$ has been introduced in [7]. The condition has been generalized by many authors to corresponding conditions for generalizations of the Cuntz-Krieger algebras, for instance, infinite directed graphs ([17]), infinite matrices with entries in $\{0,1\}$ ([11]), Hilbert C^*-bimodules ([13],see also [37], etc.). The condition (I) for C^*-symbolic dynamical systems (including λ-graph systems) has been also defined in [26](cf. [22], [23]). All of these conditions give rise to the uniqueness of the associated C^*-algebras subject to some operator relations of the generating elements.

In this section, we will introduce the notion of condition (I) for C^*-textile dynamical systems to prove the uniqueness of the C^*-algebras $\mathcal{O}^\kappa_{\rho,\eta}$ under the relation $(\rho,\eta;\kappa)$. In what follows, for a subset F of a C^*-algebra $\mathcal{B}$, we will denote by $C^*(F)$ the C^*-subalgebra of $\mathcal{B}$ generated by F.

Let $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$ be a C^*-symbolic dynamical system over Σ and $X^\kappa_{\rho,\eta}$ the associated two-dimensional subshift. Denote by $\Lambda_\rho,\Lambda_\eta$ the associated subshifts to the C^*-symbolic dynamical systems $(\mathcal{A},\rho,\Sigma^\rho),(\mathcal{A},\eta,\Sigma^\eta)$ respectively. For $\mu=(\mu_1,\dots,\mu_j)\in B_j(\Lambda_\rho),\zeta=(\zeta_1,\dots,\zeta_k)\in B_k(\Lambda_\eta)$, we put $S_\mu=S_{\mu_1}\cdots S_{\mu_j},T_\zeta=T_{\zeta_1}\cdots T_{\zeta_k}$ and $\rho_\mu=\rho_{\mu_j}\circ\cdots\circ\rho_{\mu_1},\eta_\mu=\eta_{\zeta_k}\circ\cdots\circ\eta_{\zeta_1}$ respectively. We denote by $|\mu|,|\zeta|$ the lengths j,k respectively.

In the algebra $\mathcal{O}^\kappa_{\rho,\eta}$, we set the subalgebras

$$\mathcal{F}_{\rho,\eta}=C^*(S_\mu T_\zeta x T^*_\xi S^*_\nu : \mu,\nu\in B_*(\Lambda_\rho),\zeta,\xi\in B_*(\Lambda_\eta),|\mu|=|\nu|,|\zeta|=|\xi|,x\in\mathcal{A})$$

and for $j,k\in\mathbb{Z}_+$

$$\mathcal{F}_{j,k}=C^*(S_\mu T_\zeta x T^*_\xi S^*_\nu : \mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta),x\in\mathcal{A}).$$

We notice that

$$\mathcal{F}_{j,k}=C^*(T_\zeta S_\mu x S^*_\nu T^*_\xi : \mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta),x\in\mathcal{A})$$

The identities

$$S_\mu T_\zeta x T^*_\xi S^*_\nu=\sum_{a\in\Sigma^\eta} S_\mu T_{\zeta a}\eta_a(x)T^*_{\xi a}S^*_\nu, \tag{4.1}$$

$$T_\zeta S_\mu x S^*_\nu T^*_\xi=\sum_{\alpha\in\Sigma^\rho} T_\zeta S_{\mu\alpha}\rho_\alpha(x)S^*_{\nu\alpha}T^*_\xi \tag{4.2}$$

for $x\in\mathcal{A}$ and $\mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta)$ yield the embeddings

$$\mathcal{F}_{j,k}\hookrightarrow\mathcal{F}_{j,k+1},\qquad \mathcal{F}_{j,k}\hookrightarrow\mathcal{F}_{j+1,k}$$

such that $\cup_{j,k\in\mathbb{Z}_+}\mathcal{F}_{j,k}$ is dense in $\mathcal{F}_{\rho,\eta}$.

By the universality of $\mathcal{O}^\kappa_{\rho,\eta}$, we may define an action $\widehat{\kappa}:\mathbb{T}^2\longrightarrow\mathrm{Aut}(\mathcal{O}^\kappa_{\rho,\eta})$ of the 2-dimensiona torus group $\mathbb{T}^2=\{(z,w)\in\mathbb{C}^2\mid |z|=|w|=1\}$ to $\mathcal{O}^\kappa_{\rho,\eta}$ by setting

$$\widehat{\kappa}_{z,w}(S_\alpha)=zS_\alpha,\quad \widehat{\kappa}_{z,w}(T_a)=wT_a,\quad \widehat{\kappa}_{z,w}(x)=x$$

for $\alpha\in\Sigma^\rho,a\in\Sigma^\eta$, $x\in\mathcal{A}$ and $z,w\in\mathbb{T}$. We call the action $\widehat{\kappa}:\mathbb{T}^2\longrightarrow\mathrm{Aut}(\mathcal{O}^\kappa_{\rho,\eta})$ the gauge action of $\mathbb{T}^2$ on $\mathcal{O}^\kappa_{\rho,\eta}$. The fixed point algebra of $\mathcal{O}^\kappa_{\rho,\eta}$ under $\widehat{\kappa}$ is denoted by $(\mathcal{O}^\kappa_{\rho,\eta})^{\widehat{\kappa}}$. Let $\mathcal{E}_{\rho,\eta}:\mathcal{O}^\kappa_{\rho,\eta}\longrightarrow(\mathcal{O}^\kappa_{\rho,\eta})^{\widehat{\kappa}}$ be the conditional expectaton defined by

$$\mathcal{E}_{\rho,\eta}(X)=\int_{(z,w)\in\mathbb{T}^2}\widehat{\kappa}_(z,w)(X)\,dzdw,\qquad X\in\mathcal{O}^\kappa_{\rho,\eta}.$$

The following lemma is routine.

Lemma 4.1. $(\mathcal{O}^{\kappa}_{\rho,\eta})^{\widehat{\kappa}} = \mathcal{F}_{\rho,\eta}$.

Put $\phi_\rho, \phi_\eta : \mathcal{D}_{\rho,\eta} \longrightarrow \mathcal{D}_{\rho,\eta}$ by setting

$$\phi_\rho(X) = \sum_{\alpha\in\Sigma^\rho} S_\alpha X S_\alpha^*, \qquad \phi_\eta(X) = \sum_{a\in\Sigma^\eta} T_a X T_a^*, \qquad X \in \mathcal{D}_{\rho,\eta}.$$

It is easy to see

$$\phi_\rho \circ \phi_\eta = \phi_\eta \circ \phi_\rho \quad \text{ on } \mathcal{D}_{\rho,\eta}.$$

Definition. A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies *condition* (I) if there exists a unital increasing sequence

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$$

of C^*-subalgebras of $\mathcal{A}$ such that

(1) $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$, $\eta_a(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbb{Z}_+, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$,

(2) $\cup_{l\in\mathbb{Z}_+}\mathcal{A}_l$ is dense in $\mathcal{A}$,

(3) for $\epsilon > 0$, $j, k, l \in \mathbb{N}$ with $j + k \le l$ and $X_0 \in \mathcal{F}^l_{j,k} = C^*(S_\mu T_\zeta x T_\xi^* S_\nu^* : \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}_l)$, there exists an element $g \in \mathcal{D}_{\rho,\eta} \cap \mathcal{A}_l{}'(= \{y \in \mathcal{D}_{\rho,\eta} \mid ya = ay \text{ for } a \in \mathcal{A}_l\})$ with $0 \le g \le 1$ such that

(i) $\|X_0 \phi_\rho^j \circ \phi_\eta^k(g)\| \ge \|X_0\| - \epsilon$,

(ii) $\phi_\rho^n(g)\phi_\eta^m(g) = \phi_\rho^n((\phi_\eta^m(g)))g = \phi_\rho^n(g)g = \phi_\eta^m(g)g = 0$ for all $n = 1, 2, \dots, j$, $m = 1, 2, \dots, k$.

If in particular, one may take the above subalgebras $\mathcal{A}_l \subset \mathcal{A}$, $l = 0, 1, 2, \dots$ to be of finite dimensional, then $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to satisfy *AF-condition* (I). In this case, $\mathcal{A} = \overline{\cup_{l=0}^\infty \mathcal{A}_l}$ is an AF-algebra.

As the element g above belongs to the diagonal subalgebra $\mathcal{D}_{\rho,\eta}$ of $\mathcal{F}_{\rho,\eta}$, the condition (I) of $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is intrinsically determined by itself by virtue of Lemma 4.3 below.

We will also introduce the following condition called *free*, which will be stronger than condition (I) but easier to confirm than condition (I).

Definition. A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to be *free* if there exists a unital increasing sequence

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$$

of C^*-subalgebras of $\mathcal{A}$ such that

(1) $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$, $\eta_a(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbb{Z}_+, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$,

(2) $\cup_{l\in\mathbb{Z}_+}\mathcal{A}_l$ is dense in $\mathcal{A}$,

(3) for $j, k, l \in \mathbb{N}$ with $j + k \le l$ there exists a projection $q \in \mathcal{D}_{\rho,\eta} \cap \mathcal{A}_l{}'$ such that

(i) $qa \ne 0$ for $0 \ne a \in \mathcal{A}_l$,

(ii) $\phi_\rho^n(q)\phi_\eta^m(q) = \phi_\rho^n((\phi_\eta^m(q)))q = \phi_\rho^n(q)q = \phi_\eta^m(q)q = 0$ for all $n = 1, 2, \dots, j$, $m = 1, 2, \dots, k$.

If in particular, one may take the above subalgebras $\mathcal{A}_l \subset \mathcal{A}$, $l = 0, 1, 2, \dots$ to be of finite dimensional, then $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to be *AF-free* .

Proposition 4.2. *If a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is free (resp. AF-free), then it satisfies condition (I) (resp. AF-condition (I)).*

Proof. Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is free. Take an increasing sequence $\mathcal{A}_l, l \in \mathbb{N}$ of C^*-subalgebras of $\mathcal{A}$ satisfying the above conditions (1),(2),(3) of freeness. For $j, k, l \in \mathbb{N}$ with $j + k \le l$ there exists a projection $q \in \mathcal{D}_{\rho,\eta} \cap {\mathcal{A}_l}'$ satisfying the above two conditions (i) and (ii) of (3). Put $Q^l_{j,k} = \phi^j_\rho(\phi^k_\eta(q))$. For $x \in \mathcal{A}_l, \mu, \nu \in B_j(\Lambda_\rho), \xi, \zeta \in B_k(\Lambda_\eta)$, one has the equality

$$Q^l_{j,k} S_\mu T_\zeta x T^*_\xi S^*_\nu = S_\mu T_\zeta x T^*_\xi S^*_\nu$$

so that $Q^l_{j,k}$ commutes with all of elements of $\mathcal{F}^l_{j,k}$. By using the condition (i) of (3) for q one directly sees that $S_\mu T_\zeta x T^*_\xi S^*_\nu \ne 0$ if and only if $Q^l_{j,k} S_\mu T_\zeta x T^*_\xi S^*_\nu \ne 0$. Hence the map

$$X \in \mathcal{F}^l_{j,k} \longrightarrow X Q^l_{j,k} \in \mathcal{F}^l_{j,k} Q^l_{j,k}$$

defines a homomorphism, that is proved to be injective by a similar proof to the proof of [27, Proposition 3.7]. Hence we have $\|X Q^l_{j,k}\| = \|X\| \ge \|X\| - \epsilon$ for all $X \in \mathcal{F}^l_{j,k}$. □

Let $\mathcal{B}$ be a unital C^*-algebra. Suppose that there exist an injective $*$-homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving their units and two families $s_\alpha \in \mathcal{B}, \alpha \in \Sigma^\rho$ and $t_a \in \mathcal{B}, a \in \Sigma^\eta$ of partial isometries satisfying

$$\sum_{\beta \in \Sigma^\rho} s_\beta s^*_\beta = 1, \qquad \pi(x) s_\alpha s^*_\alpha = s_\alpha s^*_\alpha \pi(x), \qquad s^*_\alpha \pi(x) s_\alpha = \pi(\rho_\alpha(x)),$$
$$\sum_{b \in \Sigma^\eta} t_b t^*_b = 1, \qquad \pi(x) t_a t^*_a = t_a t^*_a \pi(x), \qquad t^*_a \pi(x) t_a = \pi(\eta_a(x)),$$
$$s_\alpha t_b = t_a s_\beta \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$. Put $\widetilde{\mathcal{A}} = \pi(\mathcal{A})$ and $\tilde{\rho}_\alpha(\pi(x)) = \pi(\rho_\alpha(x)), \tilde{\eta}_a(\pi(x)) = \pi(\eta_a(x)), x \in \mathcal{A}$. It is easy to see that $(\widetilde{\mathcal{A}}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \kappa)$ is a C^*-textile dynamical system such that the presented two-dimensional textile dynamical system $X^\kappa_{\tilde{\rho},\tilde{\eta}}$ is the same as the one $X^\kappa_{\rho,\eta}$ presented by $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$. Let $\mathcal{O}_{\pi,s,t}$ be the C^*-subalgebra of $\mathcal{B}$ generated by $\pi(x)$ and s_α, t_a for $x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$. Let $\mathcal{F}_{\pi,s,t}$ be the C^*-subalgebra of $\mathcal{O}_{\pi,s,t}$ generated by $s_\mu t_\zeta \pi(x) t^*_\xi s^*_\nu$ for $x \in \mathcal{A}$ and $\mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta)$ with $|\mu| = |\nu|, |\zeta| = |\xi|$. By the universality of the algebra $\mathcal{O}^\kappa_{\rho,\eta}$, the correspondence

$$x \in \mathcal{A} \longrightarrow \pi(x) \in \widetilde{A}, \qquad S_\alpha \longrightarrow s_\alpha, \ \ \alpha \in \Sigma^\rho, \qquad T_a \longrightarrow t_a, \ \ a \in \Sigma^\eta$$

extends to a surjective $*$-homomorphism $\tilde{\pi} : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow \mathcal{O}_{\pi,s,t}$.

Lemma 4.3. *The restriction of $\tilde{\pi}$ to the subalgebra $\mathcal{F}_{\rho,\eta}$ is a $*$-isomorphism from $\mathcal{F}_{\rho,\eta}$ to $\mathcal{F}_{\pi,s,t}$. Hence if $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I), so does $(\widetilde{\mathcal{A}}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \kappa)$.*

Proof. It suffices to show that $\tilde{\pi}$ is injective on $\mathcal{F}_{j,k}$ for all $j, k \in \mathbb{Z}$. Suppose

$$\sum_{\mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta)} s_\mu t_\zeta \pi(x_{\mu,\zeta,\xi,\nu}) t^*_\xi s^*_\nu = 0$$

for $\sum_{\mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta)} S_\mu T_\zeta x_{\mu,\zeta,\xi,\nu} T_\xi^* S_\nu^* \in \mathcal{F}_{j,k}$ with $x_{\mu,\zeta,\xi,\nu} \in \mathcal{A}$. For $\mu',\nu' \in B_j(\Lambda_\rho), \zeta',\xi' \in B_k(\Lambda_\eta)$, one has

$$\begin{aligned}&\pi(\eta_{\zeta'}(\rho_{\mu'}(1))x_{\mu',\zeta',\xi',\nu'}\eta_{\xi'}(\rho_{\nu'}(1)))\\ =&t_{\zeta'}^* s_{\mu'}^* (\sum_{\mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta)} s_\mu t_\zeta \pi(x_{\mu,\zeta,\xi,\nu}) t_\xi^* s_\nu^*) s_{\nu'} t_{\xi'} = 0.\end{aligned}$$

As $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ is injective, one sees

$$\eta_{\zeta'}(\rho_{\mu'}(1))x_{\mu',\zeta',\xi',\nu'}\eta_{\xi'}(\rho_{\nu'}(1)) = 0$$

so that

$$S_{\mu'} T_{\zeta'} x_{\mu',\zeta',\xi',\nu'} T_{\xi'}^* S_{\nu'}^* = 0.$$

Hence we have

$$\sum_{\mu,\nu\in B_j(\Lambda_\rho),\zeta,\xi\in B_k(\Lambda_\eta)} S_\mu T_\zeta x_{\mu,\zeta,\xi,\nu} T_\xi^* S_\nu^* = 0.$$

Therefore $\tilde{\pi}$ is injective on $\mathcal{F}_{j,k}$. □

We henceforth assume that $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$ satisfies condition (I) defined above. Take a unital increasing sequence $\{\mathcal{A}_l\}_{l\in\mathbb{Z}_+}$ of C^*-subalgebras of $\mathcal{A}$ as in the definition of condition (I). Recall that the algebra $\mathcal{F}_{j,k}^l$ for $j,k \le l$ is defined as

$$\mathcal{F}_{j,k}^l = C^*(S_\mu T_\zeta x T_\xi^* S_\nu^* : \mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta), x \in \mathcal{A}_l).$$

There exists an inclusion relation $\mathcal{F}_{j,k}^l \subset \mathcal{F}_{j',k'}^{l'}$ for $j \le j', k \le k'$ and $l \le l'$ through the identities (4.1), (4.2).

Let $\mathcal{P}_{\pi,s,t}$ be the $*$-subalgebra of $\mathcal{O}_{\pi,s,t}$ algebraically generated by $\pi(x), s_\alpha, t_a$ for $x \in \mathcal{A}_l, l \in \mathbb{Z}_+,\ \alpha \in \Sigma^\rho, a \in \Sigma^\eta$.

Lemma 4.4. *Any element $x \in \mathcal{P}_{\pi,s,t}$ can be expressed in a unique way as*

$$\begin{aligned} x = &\sum_{|\nu|,|\xi|\ge 1} x_{-\xi,-\nu} t_\xi^* s_\nu^* + \sum_{|\zeta|,|\nu|\ge 1} t_\zeta x_{\zeta,-\nu} s_\nu^* + \sum_{|\mu|,|\xi|\ge 1} s_\mu x_{\mu,-\xi} t_\xi^* + \sum_{|\mu|,\zeta|\ge 1} s_\mu t_\zeta x_{\mu,\zeta}\\ &+ \sum_{|\xi|\ge 1} x_{-\xi} t_\xi^* + \sum_{|\nu|\ge 1} x_{-\nu} s_\nu^* + \sum_{|\mu|\ge 1} s_\mu x_\mu + \sum_{|\zeta|\ge 1} t_\zeta x_\zeta + x_0\end{aligned}$$

where $x_{-\xi,-\nu}, x_{\zeta,-\nu}, x_{\mu,-\xi}, x_{\mu,\zeta}, x_{-\xi}, x_{-\nu}, x_\mu, x_\zeta, x_0 \in \mathcal{P}_{\pi,s,t} \cap \mathcal{F}_{\pi,s,t}$ for $\mu,\nu \in B_*(\Lambda_\rho), \zeta,\xi \in B_*(\Lambda_\eta)$, which satisfy

$$\begin{aligned}&x_{-\xi,-\nu} = x_{-\xi,-\nu}\eta_\xi(\rho_\nu(1)), \quad x_{\zeta,-\nu} = \eta_\zeta(1)x_{\zeta,-\nu}\rho_\nu(1),\\ &x_{\mu,-\xi} = \rho_\mu(1)x_{\mu,-\xi}\eta_\xi(1), \quad x_{\mu,\zeta} = \eta_\zeta(\rho_\mu(1))x_{\mu,\zeta},\\ &x_{-\xi} = x_{-\xi}\eta_\xi(1), \quad x_{-\nu} = x_{-\nu}\rho_\nu(1), \quad x_\mu = \rho_\mu(1)x_\mu, \quad x_\zeta = \eta_\zeta(1)x_\zeta.\end{aligned}$$

Proof. Put

$$\begin{aligned}&x_{-\xi,-\nu} = \mathcal{E}_{\rho,\eta}(x s_\nu t_\xi), \quad x_{\zeta,-\nu} = \mathcal{E}_{\rho,\eta}(t_\zeta^* x s_\nu),\\ &x_{\mu,-\xi} = \mathcal{E}_{\rho,\eta}(s_\mu^* x t_\xi), \quad x_{\mu,\zeta} = \mathcal{E}_{\rho,\eta}(t_\zeta^* s_\mu^* x),\\ &x_{-\xi} = \mathcal{E}_{\rho,\eta}(x t_\xi), \quad x_{-\nu} = \mathcal{E}_{\rho,\eta}(x s_\nu), \quad x_\mu = \mathcal{E}_{\rho,\eta}(s_\mu^* x), \quad x_\zeta = \mathcal{E}_{\rho,\eta}(t_\zeta^* x),\\ &x_0 = \mathcal{E}_{\rho,\eta}(x).\end{aligned}$$

Then we have a desired expression of x. □

Lemma 4.5. *For $h \in \mathcal{D}_{\rho,\eta} \cap \mathcal{A}_l'$ and $j,k \in \mathbb{Z}$ with $j+k \le l$, put $h^{j,k} = \phi_\rho^j \circ \phi_\eta^k(h)$. Then we have*

(i) $h^{j,k}s_\mu = s_\mu h^{j-|\mu|,k}$ *for* $\mu \in B_*(\Lambda_\rho)$ *with* $|\mu| \le j$.
(ii) $h^{j,k}t_\zeta = t_\zeta h^{j,k-|\zeta|}$ *for* $\zeta \in B_*(\Lambda_\eta)$ *with* $|\zeta| \le k$.
(iii) $h^{j,k}$ *commutes with any element of* $\mathcal{F}^l_{j,k}$.

Proof. (i) It follows that for $\mu \in B_*(\Lambda_\rho)$ with $|\mu| \le j$

$$h^{j,k}s_\mu = \sum_{|\mu'|=|\mu|} s_{\mu'}\phi_\rho^{j-|\mu|}(\phi_\eta^k(h))s_{\mu'}^* s_\mu = s_\mu \phi_\rho^{j-|\mu|}(\phi_\eta^k(h))s_\mu^* s_\mu.$$

Since $h \in \mathcal{A}_l'$ and $\mathcal{A}_{j+k} \subset \mathcal{A}_l$, one has

$$\begin{aligned}
\phi_\rho^{j-|\mu|}(\phi_\eta^k(h))s_\mu^* s_\mu &= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu t_\xi h t_\xi^* s_\nu^* s_\mu^* s_\mu \\
&= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu t_\xi h t_\xi^* s_\nu^* s_\mu^* s_\mu s_\nu t_\xi t_\xi^* s_\nu^* \\
&= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu t_\xi h \eta_\xi(\rho_{\mu\nu}(1)) t_\xi^* s_\nu^* \\
&= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu t_\xi \eta_\xi(\rho_{\mu\nu}(1)) h t_\xi^* s_\nu^* \\
&= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu \rho_{\mu\nu}(1) t_\xi h t_\xi^* s_\nu^* \\
&= s_\mu^* s_\mu \phi_\rho^{j-|\mu|}(\phi_\eta^k(h)) = s_\mu^* s_\mu h^{j-|\mu|,k}
\end{aligned}$$

so that $h^{j,k}s_\mu = s_\mu h^{j-|\mu|,k}$.

(ii) Similarly we have $h^{j,k}t_\zeta = t_\zeta h^{j,k-|\zeta|}$ for $\zeta \in B_*(\Lambda_\eta)$ with $|\zeta| \le k$.

(iii) For $x \in \mathcal{A}_l, \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta)$, we have

$$h^{j,k}s_\mu t_\zeta = s_\mu h^{0,k} t_\zeta = s_\mu t_\zeta h^{0,0} = s_\mu t_\zeta h.$$

It follows that

$$h^{j,k}s_\mu t_\zeta x t_\xi^* s_\nu^* = s_\mu t_\zeta h x t_\xi^* s_\nu^* = s_\mu t_\zeta x h t_\xi^* s_\nu^* = s_\mu t_\zeta x t_\xi^* s_\nu^* h^{j,k}.$$

Hence $h^{j,k}$ commutes with any element of $\mathcal{F}^l_{j,k}$. $\square$

Lemma 4.6. *Assume that* $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ *satisfies condition (I). Let* $x \in \mathcal{P}_{\pi,s,t}$ *be expressed as in the preceding lemma. Then we have*

$$\|x_0\| \le \|x\|.$$

Proof. We may assume that for $x \in \mathcal{P}_{\pi,s,t}$,

$$x_{-\xi,-\nu}, x_{\zeta,-\nu}, x_{\mu,-\xi}, x_{\mu,\zeta}, x_{-\xi}, x_{-\nu}, x_\mu, x_\zeta, x_0 \in \tilde{\pi}(\mathcal{F}^{l_1}_{j_1,k_1})$$

for some j_1, k_1, l_1 and $\mu, \nu \in \cup_{n=0}^{j_0} B_n(\Lambda_\rho)$, $\zeta, \xi \in \cup_{n=0}^{k_0} B_n(\Lambda_\eta)$ for some j_0, k_0. Take $j, k, l \in \mathbb{Z}_+$ such as

$$j \ge j_0 + j_1, \qquad k \ge k_0 + k_1, \qquad l \ge \max\{j+k, l_1\}.$$

By Lemma 4.1, $(\widetilde{\mathcal{A}}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). For any $\epsilon > 0$ and the numbers j, k, l, the element $x_0 \in \tilde{\pi}(\mathcal{F}^{l_1}_{j_1,k_1})$, one may find $g \in \tilde{\pi}(\mathcal{D}_{\rho,\eta}) \cap \pi(\mathcal{A}_l)'$ with $0 \le g \le 1$ such that

(i) $\|x_0 \phi_\rho^j \circ \phi_\eta^k(g)\| \ge \|x_0\| - \epsilon$,

(ii) $\phi_\rho^n(g)\phi_\eta^m(g) = \phi_\rho^n((\phi_\eta^m(g)))g = \phi_\rho^n(g)g = \phi_\eta^m(g)g = 0$ for all $n = 1, 2, \ldots, j$, $m = 1, 2, \ldots, k$.

Put $h = g^{\frac{1}{2}}$ and $h^{j,k} = \phi_\rho^j \circ \phi_\eta^k(h)$. It follows that

$$\begin{aligned}
\|x\| \geq & \|h^{j,k} x h^{j,k}\| \\
= & \| \sum_{|\nu|,|\xi|\geq 1} h^{j,k} x_{-\xi,-\nu} t_\xi^* s_\nu^* h^{j,k} \qquad (1) \\
& + \sum_{|\zeta|,|\nu|\geq 1} h^{j,k} t_\zeta x_{\zeta,-\nu} s_\nu^* h^{j,k} \qquad (2) \\
& + \sum_{|\mu|,|\xi|\geq 1} h^{j,k} s_\mu x_{\mu,-\xi} t_\xi^* h^{j,k} \qquad (3) \\
& + \sum_{|\mu|,\zeta|\geq 1} h^{j,k} s_\mu t_\zeta x_{\mu,\zeta} h^{j,k} \qquad (4) \\
& + \sum_{|\xi|\geq 1} h^{j,k} x_{-\xi} t_\xi^* h^{j,k} + \sum_{|\nu|\geq 1} h^{j,k} x_{-\nu} s_\nu^* h^{j,k} + \sum_{|\mu|\geq 1} h^{j,k} s_\mu x_\mu h^{j,k} + \sum_{|\zeta|\geq 1} h^{j,k} t_\zeta x_\zeta h^{j,k} \qquad (5) \\
& + h^{j,k} x_0 h^{j,k}\|
\end{aligned}$$

For (1), as $x_{-\xi,-\nu} \in \tilde{\pi}(\mathcal{F}_{j_1,k_1}^{l_1}) \subset \tilde{\pi}(\mathcal{F}_{j,k}^{l})$, one sees that $x_{-\xi,-\nu}$ commutes with $h^{j,k}$. Hence we have

$$h^{j,k} x_{-\xi,-\nu} t_\xi^* s_\nu^* h^{j,k} = x_{-\xi,-\nu} h^{j,k} t_\xi^* s_\nu^* h^{j,k} = x_{-\xi,-\nu} h^{j,k} h^{j-|\nu|,k-|\xi|} t_\xi^* s_\nu^*$$

and

$$\begin{aligned}
h^{j,k} h^{j-|\nu|,k-|\xi|} (h^{j,k} h^{j-|\nu|,k-|\xi|})^* = & \phi_\rho^j(\phi_\eta^k(g)) \cdot \phi_\rho^{j-|\nu|}(\phi_\eta^{k-|\xi|}(g)) \\
= & \phi_\rho^{j-|\nu|} \circ \phi_\eta^{k-|\xi|}(\phi_\eta^{|\xi|}(\phi_\rho^{|\nu|}(g)g)) = 0
\end{aligned}$$

so that

$$h^{j,k} x_{-\xi,-\nu} t_\xi^* s_\nu^* h^{j,k} = 0.$$

For (2), as $x_{\xi,-\nu} \in \tilde{\pi}(\mathcal{F}_{j_1,k_1}^{l_1}) \subset \tilde{\pi}(\mathcal{F}_{j,k-|\xi|}^{l})$, one sees $x_{\xi,-\nu}$ that commutes with $h^{j,k-|\xi|}$. Hence we have

$$h^{j,k} t_\xi x_{\xi,-\nu} s_\nu^* h^{j,k} = t_\xi h^{j,k-|\xi|} x_{\xi,-\nu} h^{j-|\nu|,k} s_\nu^* = t_\xi x_{\xi,-\nu} h^{j,k-|\xi|} h^{j-|\nu|,k} s_\nu^*$$

and

$$\begin{aligned}
h^{j,k-|\xi|} h^{j-|\nu|,k} (h^{j,k-|\xi|} h^{j-|\nu|,k})^* = & \phi_\rho^j(\phi_\eta^{k-|\zeta|}(g)) \cdot \phi_\rho^{j-|\nu|}(\phi_\eta^k(g)) \\
= & \phi_\rho^{j-|\nu|} \circ \phi_\eta^{k-|\zeta|}(\phi_\rho^{|\nu|}(g)\phi_\eta^{|\zeta|}(g)) = 0
\end{aligned}$$

so that

$$h^{j,k} t_\xi x_{\xi,-\nu} s_\nu^* h^{j,k} = 0.$$

For (3), as $x_{\mu,-\xi} \in \tilde{\pi}(\mathcal{F}_{j_1,k_1}^{l_1}) \subset \tilde{\pi}(\mathcal{F}_{j-|\mu|,k}^{l})$, one sees $x_{\mu,-\xi}$ that commutes with $h^{j-|\mu|,k}$. Hence we have

$$h^{j,k} s_\mu x_{\mu,-\xi} t_\xi^* h^{j,k} = s_\mu h^{j-|\mu|,k} x_{\mu,-\xi} h^{j,k-|\xi|} t_\xi^* = s_\mu x_{\mu,-\xi} h^{j-|\mu|,k} h^{j,k-|\xi|} t_\xi^*$$

and

$$\begin{aligned}
h^{j-|\mu|,k} h^{j,k-|\xi|} (h^{j-|\mu|,k} h^{j,k-|\xi|})^* = & \phi_\rho^{j-|\mu|}(\phi_\eta^k(g)) \cdot \phi_\rho^j(\phi_\eta^{k-|\xi|}(g)) \\
= & \phi_\rho^{j-|\mu|} \circ \phi_\eta^{k-|\xi|}(\phi_\eta^{|\xi|}(g)\phi_\rho^{|\mu|}(g)) = 0
\end{aligned}$$

so that

$$h^{j,k} s_\mu x_{\mu,-\xi} t_\xi^* h^{j,k} = 0.$$

For (4), as $x_{\mu,\zeta} \in \tilde{\pi}(\mathcal{F}_{j_1,k_1}^{l_1}) \subset \tilde{\pi}(\mathcal{F}_{j-|\mu|,k-|\zeta|}^{l})$, one sees $x_{\mu,\zeta}$ that commutes with $h^{j-|\mu|,k-|\zeta|}$. Hence we have

$$h^{j,k} s_\mu t_\zeta x_{\mu,\zeta} h^{j,k} = s_\mu t_\zeta h^{j-|\mu|,k-|\zeta|} x_{\mu,\zeta} h^{j,k} = s_\mu t_\zeta x_{\mu,\zeta} h^{j-|\mu|,k-|\zeta|} h^{j,k}$$

and

$$\begin{aligned} h^{j-|\mu|,k-|\zeta|} h^{j,k} (h^{j-|\mu|,k-|\zeta|} h^{j,k})^* =& \phi_\rho^{j-|\mu|}(\phi_\eta^{k-|\zeta|}(g)) \phi_\rho^j(\phi_\eta^k(g)) \\ =& \phi_\rho^{j-|\mu|} \circ \phi_\eta^{k-|\xi|}(g \phi_\rho^{|\mu|} \circ \phi_\eta^{|\xi|}(g)) = 0 \end{aligned}$$

so that

$$h^{j,k} s_\mu t_\zeta x_{\mu,\zeta} h^{j,k} = 0.$$

For (5) as $x_{-\xi}$ commutes with $h^{j,k}$, we have

$$h^{j,k} x_{-\xi} t_\xi^* h^{j,k} = x_{-\xi} h^{j,k} h^{j,k-|\xi|} t_\xi^*$$

and

$$\begin{aligned} h^{j,k} h^{j,k-|\xi|} (h^{j,k} h^{j,k-|\xi|})^* =& \phi_\rho^j(\phi_\eta^{k|}(g)) \phi_\rho^j(\phi_\eta^{k-|\xi|}(g)) \\ =& \phi_\rho^j \circ \phi_\eta^{k-|\xi|}(\phi_\eta^{|\xi|}(g)) = 0 \end{aligned}$$

so that

$$h^{j,k} x_{-\xi} t_\xi^* h^{j,k} = 0.$$

We similarly see that

$$h^{j,k} x_{-\nu} s_\nu^* h^{j,k} = h^{j,k} s_\mu x_\mu h^{j,k} = h^{j,k} t_\zeta x_\zeta h^{j,k} = 0.$$

Therefore we have

$$\|x\| \geq \|h^{j,k} x_0 h^{j,k}\| = \|x_0 (h^{j,k})^2\| = \|x_0 \phi_\rho^j \circ \phi_\eta^k(g)\| \geq \|x_0\| - \epsilon.$$

Hence we get $\|x\| \geq \|x_0\|$. $\square$

By a similar argument ot [7, 2.8 Proposition], one sees

Corollary 4.7. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). There exists a conditional expectation $\mathcal{E}_{\pi,s,t} : \mathcal{O}_{\pi,s,t} \longrightarrow \mathcal{F}_{\pi,s,t}$ such that $\mathcal{E}_{\pi,s,t} \circ \tilde{\pi} = \tilde{\pi} \circ \mathcal{E}_{\rho,\eta}$.*

Therefore we have

Proposition 4.8. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). The $*$-homomorphism $\tilde{\pi} : \mathcal{O}_{\rho,\eta}^\kappa \longrightarrow \mathcal{O}_{\pi,s,t}$ defined by*

$$\tilde{\pi}(x) = \pi(x), \quad x \in \mathcal{A}, \qquad \tilde{\pi}(S_\alpha) = s_\alpha, \quad \alpha \in \Sigma^\rho, \qquad \tilde{\pi}(T_a) = t_a, \quad a \in \Sigma^\eta$$

becomes a surjective $*$-isomorphism, and hence the C^*-algebras $\mathcal{O}_{\rho,\eta}^\kappa$ and $\mathcal{O}_{\pi,s}$ are canonically $*$-isomorphic through $\tilde{\pi}$.

Proof. The map $\tilde{\pi} : \mathcal{F}_{\rho,\eta} \to \mathcal{F}_{\pi,s,t}$ is $*$-isomorphic and satisfies $\mathcal{E}_{\pi,s,t} \circ \tilde{\pi} = \tilde{\pi} \circ \mathcal{E}_{\rho,\eta}$. Since $\mathcal{E}_\rho : \mathcal{O}_{\rho,\eta}^\kappa \longrightarrow \mathcal{F}_{\rho,\eta}$ is faithful, a routine argument shows that the $*$-homomorphism $\tilde{\pi} : \mathcal{O}_{\rho,\eta}^\kappa \longrightarrow \mathcal{O}_{\pi,s,t}$ is actually a $*$-isomorphism. $\square$

Hence the following uniqueness of the C^*-algebra $\mathcal{O}_{\rho,\eta}^\kappa$ holds.

Theorem 4.9. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). The C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ is the unique C^*-algebra subject to the relation $(\rho, \eta; \kappa)$. This means that if there exist a unital C^*-algebra $\mathcal{B}$ and an injective $*$-homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ and two families of partial isometries $s_\alpha, \alpha \in \Sigma^\rho$, $t_a, a \in \Sigma^\eta$ satisfying the following relations :*

$$\sum_{\beta \in \Sigma^\rho} s_\beta s_\beta^* = 1, \qquad \pi(x) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi(x), \qquad s_\alpha^* \pi(x) s_\alpha = \pi(\rho_\alpha(x)),$$

$$\sum_{b \in \Sigma^\eta} t_b t_b^* = 1, \qquad \pi(x) t_a t_a^* = t_a t_a^* \pi(x), \qquad t_a^* \pi(x) t_a = \pi(\eta_a(x))$$

$$s_\alpha t_b = t_a s_\beta \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta)$$

for $(\alpha, b) \in \Sigma^{\rho\eta}, (a, \beta) \in \Sigma^{\eta\rho}$ and $x \in \mathcal{A}$, $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, then the correspondence

$$x \in \mathcal{A} \longrightarrow \pi(x) \in \mathcal{B}, \quad S_\alpha \longrightarrow s_\alpha \in \mathcal{B}, \qquad T_a \longrightarrow t_a \in \mathcal{B}$$

extends to a $*$-isomorphism $\tilde{\pi}$ from $\mathcal{O}^\kappa_{\rho,\eta}$ onto the C^*-subalgebra $\mathcal{O}_{\pi,s,t}$ of $\mathcal{B}$ generated by $\pi(x), x \in \mathcal{A}$ and $s_\alpha, \alpha \in \Sigma$, $t_a, a \in \Sigma^\eta$.

For a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, let $\lambda_{\rho,\eta} : \mathcal{A} \to \mathcal{A}$ be the positive map on $\mathcal{A}$ defined by

$$\lambda_{\rho,\eta}(x) = \sum_{\alpha \in \Sigma^\rho, a \in \Sigma^\eta} \eta_a \circ \rho_\alpha(x), \qquad x \in \mathcal{A}.$$

Then $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to be *irreducible* if there exists no nontrivial ideal of $\mathcal{A}$ invariant under $\lambda_{\rho,\eta}$.

Corollary 4.10. *If $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I) and is irreducible, the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ is simple.*

Proof. Assume that there exists a nontrivial ideal $\mathcal{I}$ of $\mathcal{O}^\kappa_{\rho,\eta}$. Now suppose that $\mathcal{I} \cap \mathcal{A} = \{0\}$. As $S_\alpha^* S_\alpha = \rho_\alpha(1), T_a^* T_a = \eta_a(1) \in \mathcal{A}$ one knows that $S_\alpha, T_a \notin \mathcal{I}$ for all $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$. By the preceding theorem, the quotient map $q : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow \mathcal{O}^\kappa_{\rho,\eta}/\mathcal{I}$ must be injective so that $\mathcal{I}$ is trivial. Hence one sees that $\mathcal{I} \cap \mathcal{A} \neq \{0\}$ and it is invariant under $\lambda_{\rho,\eta}$. □

5. Cocrete realization

In this section we will realize the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ for $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ in a concrete way. For $\gamma_i \in \Sigma^\rho \cup \Sigma^\eta$, put

$$\xi_{\gamma_i} = \begin{cases} \rho_{\gamma_i} & \text{if } \gamma_i \in \Sigma^\rho, \\ \eta_{\gamma_i} & \text{if } \gamma_i \in \Sigma^\eta. \end{cases}$$

Definition. A finite sequence of labeles $(\gamma_1, \gamma_2, \dots, \gamma_k) \in (\Sigma^\rho \cup \Sigma^\eta)^k$ is said to be *concatenated labeled path* if $\xi_{\gamma_k} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1}(1) \neq 0$. For $m, n \in \mathbb{Z}_+$, let $L_{(n,m)}$ be the set of concatenated labeled paths $(\gamma_1, \gamma_2, \dots, \gamma_{m+n})$ such that symbols in Σ^ρ appear in $(\gamma_1, \gamma_2, \dots, \gamma_{m+n})$ n-times and symbols in Σ^η appear in $(\gamma_1, \gamma_2, \dots, \gamma_{m+n})$ m-times. We define a relation in $L_{(n,m)}$ for $i = 1, 2, \dots, n+m-1$. We write

$$(\gamma_1, \dots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{m+n}) \underset{i}{\approx} (\gamma_1, \dots, \gamma_{i-1}, \gamma'_i, \gamma'_{i+1}, \gamma_{i+2}, \dots, \gamma_{m+n})$$

if one of the following two conditions holds:
(1) $(\gamma_i, \gamma_{i+1}) \in \Sigma^{\rho\eta}, (\gamma'_i, \gamma'_{i+1}) \in \Sigma^{\eta\rho}$ and $\kappa(\gamma_i, \gamma_{i+1}) = (\gamma'_i, \gamma'_{i+1})$,
(2) $(\gamma_i, \gamma_{i+1}) \in \Sigma^{\eta\rho}, (\gamma'_i, \gamma'_{i+1}) \in \Sigma^{\rho\eta}$ and $\kappa(\gamma'_i, \gamma'_{i+1}) = (\gamma_i, \gamma_{i+1})$.

Denote by $\approx$ the equivalence relation in $L_{(n,m)}$ generated by the relations $\underset{i}{\approx}, i = 1, 2, \dots, n+m-1$. Let $\mathfrak{T}_{(n,m)} = L_{(n,m)}/\approx$ be the set of equivalence classes of $L_{(n,m)}$ under $\approx$. Denote by $[\gamma] \in \mathfrak{T}_{(n,m)}$ the equivalence class of $\gamma \in L_{(n,m)}$. Put the vectors $e = (1,0), f = (0,-1)$ in $\mathbb{R}^2$. Consider the set of all paths consisting of sequences of vectors e, f starting at the point $(-n, m) \in \mathbb{R}^2$ for $n, m \in \mathbb{Z}_+$ and ending at the origin. Such a path consists of n e-vectors and m f-vectors. Let $\mathfrak{P}_{(n,m)}$ be the set of all such paths from $(-n, m)$ to the origin. We consider the correspondence

$$\rho_\alpha \longrightarrow e \quad (\alpha \in \Sigma^\rho), \qquad \eta_a \longrightarrow f \quad (a \in \Sigma^\eta),$$

denoted by π. It extends from $L_{(n,m)}$ to $\mathfrak{P}_{(n,m)}$ in a natural way. The following lemma is obvious.

Lemma 5.1. *For any path $p \in \mathfrak{P}_{(n,m)}$ of vectors, there uniquely exists a concatenated labeled path $\gamma \in L_{(n,m)}$ such that $\pi(\gamma) = p$.*

For a concatenated labeled path $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{n+m}) \in L_{(n,m)}$, put the projection in $\mathcal{A}$

$$P_\gamma = \xi_{\gamma_k} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1}(1).$$

We note that $P_\gamma \neq 0$ for all $\gamma \in L_{(n,m)}$.

Lemma 5.2. *For $\gamma, \gamma' \in L_{(n,m)}$, if $\gamma \approx \gamma'$, we have $P_\gamma = P_{\gamma'}$. Hence the projection $P_{[\gamma]}$ for $[\gamma] \in \mathfrak{T}_{(n,m)}$ is well-defined.*

Proof. If $\kappa(\alpha, b) = (a, \beta)$, one has $\eta_b \circ \rho_\alpha(1) = \rho_\beta \circ \eta_a(1) \neq 0$. Hence we know the assertion. $\square$

Denote by $|\mathfrak{T}_{(n,m)}|$ the cardinal number of the finite set $\mathfrak{T}_{(n,m)}$. Let $e_t, t \in \mathfrak{T}_{(n,m)}$ be the standard complete orthonomal basis of $\mathbb{C}^{|\mathfrak{T}_{(n,m)}|}$. Define

$$\begin{aligned} H_{(n,m)} &= \sum_{t \in \mathfrak{T}_{(n,m)}} {}^{\oplus} \mathbb{C} e_t \otimes P_t \mathcal{A} \\ (&= \sum_{t \in \mathfrak{T}_{(n,m)}} {}^{\oplus} \mathrm{Span}\{ c e_t \otimes P_t x \mid c \in \mathbb{C}, x \in \mathcal{A}\}) \end{aligned}$$

the direct sum of $\mathbb{C}e_t \otimes P_t\mathcal{A}$ over $t \in \mathfrak{T}_{(n,m)}$. $H_{(n,m)}$ has a structure of Hilbert C^*-bimodule over $\mathcal{A}$ by setting

$$\begin{aligned} (e_t \otimes P_t x) y &:= e_t \otimes P_t x y, \\ \phi(y)(e_t \otimes P_t x) &:= e_t \otimes \xi_\gamma(y) x (= e_t \otimes P_t \xi_\gamma(y) x), \end{aligned}$$

where $t = [\gamma]$ for $\gamma = (\gamma_1, \dots, \gamma_{n+m})$ and $\xi_\gamma(y) = \xi_{\gamma_{n+m}} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1}(y)$ and

$$\langle e_t \otimes P_t x \mid e_s \otimes P_s y \rangle := \begin{cases} x^* P_t y & \text{if } t = s, \\ 0 & \text{otherwise} \end{cases}$$

for $t, s \in \mathfrak{T}_{(n,m)}$ and $x, y \in \mathcal{A}$. Put $H_{(0,0)} = \mathcal{A}$. Denote by $F(\rho, \eta)$ the Hilbert C^*-bimodule over $\mathcal{A}$ defined by the direct sum:

$$F(\rho, \eta) = \sum_{(n,m) \in \mathbb{Z}^2} {}^{\oplus} H_{(n,m)}.$$

For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, the creation operators s_α, t_a on $F(\rho, \eta)$:

$$s_\alpha : H_{(n,m)} \longrightarrow H_{(n+1,m)}, \qquad t_a : H_{(n,m)} \longrightarrow H_{(n,m+1)}$$

are defined by

$$
\begin{aligned}
s_\alpha x &= e_\alpha \otimes P_\alpha x, \qquad \text{for } x \in H_{(0,0)}(=\mathcal{A}),\\
s_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}x) &= \begin{cases} e_{[\alpha\gamma]} \otimes P_{[\alpha\gamma]}x & \text{if } \alpha\gamma \in L_{(n+1,m)},\\ 0 & \text{otherwise,}\end{cases}\\
t_a x &= e_a \otimes P_a x, \qquad \text{for } x \in H_{(0,0)}(=\mathcal{A}),\\
t_a(e_{[\gamma]} \otimes P_{[\gamma]}x) &= \begin{cases} e_{[a\gamma]} \otimes P_{[a\gamma]}x & \text{if } a\gamma \in L_{(n,m+1)},\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

For $y \in \mathcal{A}$ an operator $i_{F(\rho,\eta)}(y)$ on $F(\rho,\eta)$:

$$i_{F(\rho,\eta)}(y) : H_{(n,m)} \longrightarrow H_{(n,m)}$$

is defined by

$$
\begin{aligned}
i_{F(\rho,\eta)}(y)x &= yx \qquad \text{for } x \in H_{(0,0)}(=\mathcal{A}),\\
i_{F(\rho,\eta)}(y)(e_{[\gamma]} \otimes P_{[\gamma]}x) &= \phi(y)(e_{[\gamma]} \otimes P_{[\gamma]}x)(= e_{[\gamma]} \otimes \xi_\gamma(y)x).
\end{aligned}
$$

Define the Cuntz-Toeplitz C^*-algebra for $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$

$$\mathcal{T}^\kappa_{(\rho,\eta)} = C^*(s_\alpha, t_a, i_{F(\rho,\eta)}(y) \mid \alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A})$$

as the C^*-algebra on $F_{\rho,\eta}$ generated by $s_\alpha, t_a, i_{F(\rho,\eta)}(y)$ for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A}$.

Lemma 5.3. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have*

(i) $s^*_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} \phi(\rho_\alpha(1))(e_{[\gamma']} \otimes P_{[\gamma']}x) & \textit{if } \gamma \approx \alpha\gamma',\\ 0 & \textit{otherwise.}\end{cases}$

(ii) $t^*_a(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} \phi(\eta_a(1))(e_{[\gamma']} \otimes P_{[\gamma']}x) & \textit{if } \gamma \approx a\gamma',\\ 0 & \textit{otherwise.}\end{cases}$

Proof. (i) Suppose that $\gamma \approx \alpha\gamma'$.

$$
\begin{aligned}
\langle s^*_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}x) \mid e_{[\gamma']} \otimes P_{[\gamma']}x'\rangle &= \langle e_{[\gamma]} \otimes P_{[\gamma]}x \mid e_{[\alpha\gamma']} \otimes P_{[\alpha\gamma']}x'\rangle\\
&= \begin{cases} x^*P_{[\alpha\gamma']}x & \text{if } \gamma \approx \alpha\gamma',\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

On the other hand,

$$\phi(\rho_\alpha(1))(e_{[\gamma']} \otimes P_{[\gamma']}x) = e_{[\gamma']} \otimes P_{[\alpha\gamma']}P_{\gamma'}x = e_{[\gamma']} \otimes P_{[\alpha\gamma']}x$$

so that

$$\langle \phi(\rho_\alpha(1))(e_{[\gamma']} \otimes P_{[\gamma']}x) \mid e_{[\gamma']} \otimes P_{[\gamma']}x'\rangle = x^*P_{[\alpha\gamma']}x'.$$

Hence we obtain the desired equality. Similarly we see (ii). □

Lemma 5.4. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have*

(i) $s^*_\alpha s_\alpha = \phi(\rho_\alpha(1))$ *and*

$$s_\alpha s^*_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} e_{[\gamma]} \otimes P_{[\gamma]}x) & \textit{if } \gamma \approx \alpha\gamma' \textit{ for some } \gamma',\\ 0 & \textit{otherwise.}\end{cases}$$

(ii) $t_a^* t_a = \phi(\eta_a(1))$ *and*

$$t_a t_a^*(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} e_{[\gamma]} \otimes P_{[\gamma]}x) & \text{if } \gamma \approx a\gamma' \text{ for some } \gamma', \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) It follows that for $\gamma \in L(n,m)$

$$s_\alpha^* s_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} \phi(\rho_\alpha(1))(e_{[\gamma]} \otimes P_{[\gamma]}x) & \text{if } \alpha\gamma \in L_{(n+1,m)}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\phi(\rho_\alpha(1))(e_{[\gamma]} \otimes P_{[\gamma]}x) = e_{[\gamma]} \otimes P_{[\alpha\gamma]}P_{[\gamma]}x = e_{[\gamma]} \otimes P_{[\alpha\gamma]}x.$$

Hence $\phi(\rho_\alpha(1))(e_{[\gamma]} \otimes P_{[\gamma]}x) = 0$ if $\alpha\gamma \notin L_{(n+1,m)}$. Therefore we have

$$s_\alpha^* s_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}x) = \phi(\rho_\alpha(1))(e_{[\gamma]} \otimes P_{[\gamma]}x)$$

and hence $s_\alpha^* s_\alpha = \phi(\rho_\alpha(1))$.

The equality

$$s_\alpha s_\alpha^*(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} e_{[\gamma]} \otimes P_{[\gamma]}x & \text{if } \gamma \approx \alpha\gamma' \text{ for some } \gamma', \\ 0 & \text{otherwise} \end{cases}$$

is direct.

(ii) The assertion is similar to (i). $\square$

Lemma 5.5.

(i) $1 - \sum_{\alpha\in\Sigma^\rho} s_\alpha s_\alpha^* =$ *the projection onto the subspace spanned by the vectors* $e_{[\gamma]} \otimes P_{[\gamma]}x$ *for* $\gamma \in \cup_{m=0}^\infty L_{(0,m)}, x \in \mathcal{A}$.

(ii) $1 - \sum_{a\in\Sigma^\eta} t_a t_a^* =$ *the projection onto the subspace spanned by the vectors* $e_{[\gamma]} \otimes P_{[\gamma]}x$ *for* $\gamma \in \cup_{n=0}^\infty L_{(n,0)}, x \in \mathcal{A}$.

Lemma 5.6. *For* $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ *and* $x \in \mathcal{A}$, *we have*

(i) $s_\alpha^* x s_\alpha = \phi(\rho_\alpha(x))$.

(ii) $t_a^* x t_a = \phi(\eta_a(x))$.

Proof. For $y \in \mathcal{A}$, we have

(i)

$$\begin{aligned} s_\alpha^* x s_\alpha(e_{[\gamma]} \otimes P_\gamma y) &= s_\alpha^*(e_{[\alpha\gamma]} \otimes P_{\alpha\gamma} y \xi_{\alpha\gamma}(x)) \\ &= e_{[\gamma]} \otimes P_\gamma y \xi_\gamma(\rho_\alpha(x))) \\ &= \phi(\rho_\alpha(x))(e_{[\gamma]} \otimes P_\gamma y). \end{aligned}$$

(ii)

$$\begin{aligned} t_a^* x t_a(e_{[\gamma]} \otimes P_\gamma y) &= t_a^*(e_{[a\gamma]} \otimes P_{a\gamma} y \xi_{\alpha\gamma}(x)) \\ &= e_{[\gamma]} \otimes P_\gamma y \xi_\gamma(\eta_a(x))) \\ &= \phi(\eta_a(x))(e_{[\gamma]} \otimes P_\gamma y). \end{aligned}$$

$\square$

Lemma 5.7. *For* $\alpha, \beta \in \Sigma^\rho$, $a, b \in \Sigma^\eta$ *we have*

$$s_\alpha t_b = t_a s_\beta \qquad \text{if } \kappa(\alpha, b) = (a, \beta). \tag{5.1}$$

Proof. For $\gamma \in L_{(n,m)}$, suppose that $\alpha b\gamma, a\beta\gamma \in L_{(n+1,m+1)}$. It follows that

$$
\begin{aligned}
s_\alpha t_b(e_{[\gamma]} \otimes P_\gamma x) &= e_{[\alpha b\gamma]} \otimes P_{\alpha b\gamma} y), \\
t_a s_\beta(e_{[\gamma]} \otimes P_\gamma x) &= (e_{[a\beta\gamma]} \otimes P_{a\beta\gamma} x).
\end{aligned}
$$

Since $\kappa(\alpha, b) = (a, \beta)$, the condition $\alpha b\gamma \in L_{(n+1,m+1)}$ is equivalent to the condition $a\beta\gamma \in L_{(n+1,m+1)}$. We then have $[\alpha b\gamma] = [a\beta\gamma]$ and $P_{\alpha b\gamma} = P_{a\beta\gamma}$. $\square$

Let $\mathcal{I}_{(\rho,\eta)}$ be the ideal of $\mathcal{T}^\kappa_{(\rho,\eta)}$ generated by the projections: $1 - \sum_{\alpha\in\Sigma^\rho} s_\alpha s_\alpha^*$, and $1 - \sum_{a\in\Sigma^\eta} t_a t_a^*$. Let $\widehat{\mathcal{O}}^\kappa_{\rho,\eta}$ be the quatient C^*-algebra

$$
\widehat{\mathcal{O}}^\kappa_{\rho,\eta} = \mathcal{T}^\kappa_{(\rho,\eta)} / \mathcal{I}_{(\rho,\eta)}.
$$

Let $\pi_{(\rho,\eta)} : \mathcal{T}^\kappa_{(\rho,\eta)} \longrightarrow \widehat{\mathcal{O}}^\kappa_{\rho,\eta}$ be the quatient map. Put

$$
\widehat{S}_\alpha = \pi_{(\rho,\eta)}(s_\alpha), \quad \widehat{T}_a = \pi_{(\rho,\eta)}(t_a), \quad \hat{i}(x) = \pi_{(\rho,\eta)}(i_{(F_{(\rho,\eta)})}(x)
$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $x \in \mathcal{A}$. By the above discussions, the following relations hold:

$$
\begin{gathered}
\sum_{\beta\in\Sigma^\rho} \widehat{S}_\beta \widehat{S}_\beta^* = 1, \qquad \hat{i}(x)\widehat{S}_\alpha \widehat{S}_\alpha^* = \widehat{S}_\alpha \widehat{S}_\alpha^* \hat{i}(x), \qquad \widehat{S}_\alpha^* \hat{i}(x) \widehat{S}_\alpha = \hat{i}(\rho_\alpha(x)), \\
\sum_{b\in\Sigma^\eta} \widehat{T}_b \widehat{T}_b^* = 1, \qquad \hat{i}(x)\widehat{T}_a \widehat{T}_a^* = \widehat{T}_a \widehat{T}_a^* \hat{i}(x), \qquad \widehat{T}_a^* \hat{i}(x) \widehat{T}_a = \hat{i}(\eta_a(x)), \\
\widehat{S}_\alpha \widehat{T}_b = \widehat{T}_a \widehat{S}_\beta \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta)
\end{gathered}
$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$. Therefore we have

Proposition 5.8. *Suppose that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). Then the algebra $\widehat{\mathcal{O}}^\kappa_{\rho,\eta}$ is canonically isomorphic to the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ through the correspondences:*

$$
S_\alpha \longrightarrow \widehat{S}_\alpha, \qquad T_a \longrightarrow \widehat{T}_a, \qquad x \longrightarrow \hat{i}(x)
$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $x \in \mathcal{A}$.

6. K-Theory Machinery

In this section, we will study K-theory groups $K_*(\mathcal{O}^\kappa_{\rho,\eta})$ for the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$. We fix a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$. We define two actions

$$
\hat{\rho} : \mathbb{T} \longrightarrow \mathrm{Aut}(\mathcal{O}^\kappa_{\rho,\eta}), \quad \hat{\eta} : \mathbb{T} \longrightarrow \mathrm{Aut}(\mathcal{O}^\kappa_{\rho,\eta})
$$

of the circle group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ to $\mathcal{O}^\kappa_{\rho,\eta}$ by setting

$$
\hat{\rho}_z = \widehat{\kappa}_{(z,1)}, \qquad \hat{\eta}_w = \widehat{\kappa}_{(1,w)}, \qquad z, w \in \mathbb{T}.
$$

They satisfy

$$
\hat{\rho}_z \circ \hat{\eta}_w = \hat{\eta}_w \circ \hat{\rho}_z = \widehat{\kappa}_{(z,w)}, \qquad z, w \in \mathbb{T}.
$$

Set the fixed point algebras

$$
\begin{aligned}
(\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}} &= \{x \in \mathcal{O}^\kappa_{\rho,\eta} \mid \hat{\rho}_z(x) = x \text{ for all } z \in \mathbb{T}\}, \\
(\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\eta}} &= \{x \in \mathcal{O}^\kappa_{\rho,\eta} \mid \hat{\eta}_z(x) = x \text{ for all } z \in \mathbb{T}\}.
\end{aligned}
$$

For $x \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$, define the constant function $\widehat{x} \in L^1(\mathbb{T}, \mathcal{O}^\kappa_{\rho,\eta}) \subset \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$ by setting $\widehat{x}(z) = x, z \in \mathbb{T}$. Put $p_0 = \widehat{1}$. By [41], the algebra $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$ is canonically isomorphic to $p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0$ through the map

$$j_\rho : x \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \longrightarrow \widehat{x} \in p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0$$

which induces an isomorphism

$$j_{\rho_*} : K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \longrightarrow K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0), \qquad i = 0, 1 \tag{6.1}$$

on their K-groups.

Lemma 6.1.

(i) *There exists an isometry* $v \in M((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \otimes \mathcal{K})$ *such that* $vv^* = p_0 \otimes 1, v^*v = 1$.

(ii) $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$ *is stably isomorphic to* $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$, *and similarly* $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\eta} \mathbb{T}$ *is stably isomorphic to* $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\eta}$.

(iii) *The inclusion* $\iota_{\hat\rho} : p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0 \hookrightarrow \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$ *induces an isomorphism*

$$\iota_{\hat\rho *} : K_0(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) \cong K_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \tag{6.2}$$

on their K-groups.

Proof. (i) We will prove that p_0 is a full projection in $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$. Suppose that there exists an irreducible representation π of $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$ such that $\pi(p_0) = 0$. Denote by $*$ the $\hat\rho$-twisted convolution product in $L^1(\mathbb{T}, \mathcal{O}^\kappa_{\rho,\eta})$ (= the product in the algebra $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$). For $Y \in \mathcal{O}^\kappa_{\rho,\eta}$, put $\widehat{Y}(z) = Y$ for $z \in \mathbb{T}$. The equality $\widehat{Y} * p_0 = \widehat{Y}$ implies $\widehat{Y} \in \ker(\pi)$. For $Y, Z \in \mathcal{O}^\kappa_{\rho,\eta}$ by using the equality $\widehat{Z}^*(z) = \hat\rho_z(Z^*)$, we have $(\widehat{Y} * \widehat{Z}^*)(z) = Y\hat\rho_z(Z^*)$. For $\mu \in B_k(\Lambda_\rho)$, we have

$$(\widehat{YS_\mu} * \widehat{S}^*_\mu)(z) = z^{-k} Y S_\mu S^*_\mu$$

and hence

$$(\sum_{\mu \in B_k(\Lambda_\rho)} \widehat{YS_\mu} * \widehat{S}^*_\mu)(z) = z^{-k} Y.$$

As $\widehat{YS_\mu}, \widehat{S}^*_\mu \in \ker(\pi)$, the function $z \in \mathbb{T} \longrightarrow z^{-k} Y \in \mathcal{O}^\kappa_{\rho,\eta}$ belongs to $\ker(\pi)$ for $k = 0, 1, 2, \dots$. Let $E^l_i, i = 1, 2, \dots, m(k)$ be the minimal projections in the commutative C^*-algebra $C^*(\rho_\mu(1) \mid \mu \in B_k(\Lambda_\rho))$ generated by the projections $\rho_\mu(1), \mu \in B_k(\Lambda_\rho)$. Hence $\sum_{i=1}^{m(k)} E^k_i = 1$ and for $i = 1, \dots, m(k)$, there exists $\mu(i) \in B_k(\Lambda_\rho)$ such that $E^k_i \le S^*_{\mu(i)} S_{\mu(i)}$. Since for $Y \in \mathcal{O}^\kappa_{\rho,\eta}$,

$$(\widehat{YE^k_i S^*_\mu} * \widehat{S^*_\mu}^*)(z) = z^k Y E^k_i S^*_\mu S_\mu = z^k Y E^k_i,$$

we have

$$(\sum_{i=1}^{m(k)} \widehat{YE^k_i S^*_\mu} * \widehat{S^*_\mu}^*)(z) = z^k Y.$$

As $\widehat{YE^k_i S^*_\mu}, \widehat{S^*}^*_\mu \in \ker(\pi)$, the function $z \in \mathbb{T} \longrightarrow z^k Y \in \mathcal{O}^\kappa_{\rho,\eta}$ belongs to $\ker(\pi)$ for $k = 0, 1, 2, \dots$. Therefore we know that the functions $z \in \mathbb{T} \longrightarrow z^k Y \in \mathcal{O}^\kappa_{\rho,\eta}$ belongs to $\ker(\pi)$ for all $k \in \mathbb{Z}$. In particular, for $Y = 1$ the functions $z \in \mathbb{T} \longrightarrow z^k \in \mathcal{O}^\kappa_{\rho,\eta}$ belongs to $\ker(\pi)$ for all $k \in \mathbb{Z}$ so that $C(\mathbb{T})$ is contained in $\ker(\pi)$. Take an approximate identity $\varphi_n \in C(\mathbb{T}), n \in \mathbb{N}$ for the usual convolution product in

$L^1(\mathbb{T})$. Then for $X \in L^1(\mathbb{T}, \mathcal{O}^\kappa_{\rho,\eta})$, one has $\|X * \varphi_n - X\|_1 \to 0$ as $n \to \infty$. Since $X * \varphi_n \in \ker(\pi)$, one has $X \in \ker(\pi)$. Hence we have $L^1(\mathbb{T}, \mathcal{O}^\kappa_{\rho,\eta}) \subset \ker(\pi)$ so that $\ker(\pi) = \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$. Therefore $p_0 \in \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$ is a full projection of $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$, By [5, Corollary 2.6], there exists $v \in M((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \otimes \mathcal{K})$ such that $vv^* = p_0 \otimes 1, v^*v = 1$.

(ii) As $Ad(v^*) : x \in p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0 \otimes \mathcal{K} \to v^*xv \in \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T} \otimes \mathcal{K}$ is an isomorphism and $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$ is isomorphic to $p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0$ through j_ρ, we have $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$ is stably isomorphic to $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$.

(iii) Let $v \in M((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \otimes \mathcal{K})$ be the isometry as above such that $vv^* = p_0 \otimes 1, v^*v = 1$. For a projection $q \in p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0 \otimes \mathcal{K}$, we have $[v^*qv] = [q] = \iota_{\hat\rho *}([q])$ and hence $\iota_{\hat\rho *} = Ad(v^*)_* : K_0(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) \cong K_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})$ is an isomorphism. □

Thanks to the lemma above, $Ad(v^*) : x \in p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0 \otimes \mathcal{K} \to v^*xv \in \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T} \otimes \mathcal{K}$ induces isomorphisms

$$Ad(v^*)_* : K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) \longrightarrow K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}), \qquad i = 0, 1. \tag{6.3}$$

Let $\hat{\hat\rho}$ be the automorphism on $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$ for the positive generator of $\mathbb{Z}$ for the dual action of $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}$. By (6.1) and (6.3), we may define an isomorphism

$$\beta_{\rho,i} = j_{\rho*}^{-1} \circ Ad(v^*)_*^{-1} \circ \hat{\hat\rho}_* \circ Ad(v^*)_* \circ j_{\rho*} : K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \longrightarrow K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}), \quad i = 0, 1 \tag{6.4}$$

so that the diagram is commutative:

$$\begin{array}{ccc}
K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xrightarrow{\hat{\hat\rho}_*} & K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \\
\uparrow{\scriptstyle Ad(v^*)_*} & & \uparrow{\scriptstyle Ad(v^*)_*} \\
K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) & & K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) \\
\uparrow{\scriptstyle j_{\rho*}} & & \uparrow{\scriptstyle j_{\rho*}} \\
K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xrightarrow{\beta_{\rho,i}} & K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})
\end{array}$$

By [35] (cf. [13]), one has the six term exact sequence of K-theory:

$$\begin{array}{ccccc}
K_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xrightarrow{\mathrm{id}-\hat{\hat\rho}_*} & K_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xrightarrow{\iota_*} & K_0((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \times_{\hat{\hat\rho}} \mathbb{Z}) \\
{\scriptstyle \delta}\uparrow & & & & \downarrow{\scriptstyle \exp} \\
K_1((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \times_{\hat{\hat\rho}} \mathbb{Z}) & \xleftarrow[\iota_*]{} & K_1(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xleftarrow[\mathrm{id}-\hat{\hat\rho}_*]{} & K_1(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})
\end{array}$$

Since $(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \times_{\hat{\hat\rho}} \mathbb{Z} \cong \mathcal{O}^\kappa_{\rho,\eta} \otimes \mathcal{K}$ and $K_*(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \cong K_*((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$, one has

Lemma 6.2. *The following six term exact sequence of K-theory holds:*

$$\begin{array}{ccccc}
K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xrightarrow{\mathrm{id}-\beta_{\rho,0}} & K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xrightarrow{\iota_*} & K_0(\mathcal{O}^\kappa_{\rho,\eta}) \\
{\scriptstyle \delta}\uparrow & & & & \downarrow{\scriptstyle \exp} \\
K_1(\mathcal{O}^\kappa_{\rho,\eta}) & \xleftarrow[\iota_*]{} & K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xleftarrow[\mathrm{id}-\beta_{\rho,1}]{} & K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}).
\end{array}$$

Hence there exist short exact sequences for $i = 0, 1$:

$$
\begin{aligned}
0 &\longrightarrow \operatorname{Coker}(\mathrm{id} - \beta_{\rho,i}) \text{ in } K_i((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \\
&\longrightarrow K_i(\mathcal{O}^{\kappa}_{\rho,\eta}) \\
&\longrightarrow \operatorname{Ker}(\mathrm{id} - \beta_{\rho,i+1}) \text{ in } K_{i+1}((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \\
&\longrightarrow 0.
\end{aligned}
$$

We will then study the following groups that appear in the above sequences

$$
\operatorname{Coker}(\mathrm{id} - \beta_{\rho,i}) \text{ in } K_i((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}), \qquad \operatorname{Ker}(\mathrm{id} - \beta_{\rho,i+1}) \text{ in } K_{i+1}((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})
$$

for $i = 0, 1$. The action $\hat\eta$ acts on the subalgebra $(\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}$, which we still denote by $\hat\eta$. Then the fixed point algebra $((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta}$ of $(\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}$ under the action $\hat\eta$ coincides with $\mathcal{F}_{\rho,\eta}$. The above discussions for the action $\hat\rho : \mathbb{T} \longrightarrow \mathcal{O}^{\kappa}_{\rho,\eta}$ works for the action $\hat\eta : \mathbb{T} \longrightarrow (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}$ as in the following way. For $y \in ((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta}$, define the constant function $\widehat{y} \in L^1(\mathbb{T}, (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \subset (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$ by setting $\widehat{y}(z) = y, z \in \mathbb{T}$. Putting $q_0 = \widehat{1}$, the algebra $((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta}$ is canonically isomorphic to $q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0$ through the map

$$
j^{\rho}_{\eta} : y \in ((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta} \longrightarrow \hat{y} \in q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0
$$

which induces an isomorphism

$$
j^{\rho}_{\eta *} : K_i(((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta}) \longrightarrow K_i(q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \tag{6.5}
$$

on their K-groups. Similarly to Lemma 6.1, we have

Lemma 6.3.

(i) *There exists an isometry $u \in M(((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}) \otimes \mathcal{K})$ such that $uu^* = q_0 \otimes 1, u^*u = 1$.*

(ii) *$(\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$ is stably isomorphic to $((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta}$.*

(iii) *The inclusion $\iota^{\hat\rho}_{\hat\eta} : q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0 (= ((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta} = \mathcal{F}_{\rho,\eta}) \hookrightarrow (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$ induces an isomorphism*

$$
\iota^{\hat\rho}_{\hat\eta *} : K_0(q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \cong K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}) \tag{6.6}
$$

on their K-groups.

The isomorphism

$$
Ad(u^*) : y \in q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0 \longrightarrow u^*yu \in (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}
$$

induces isomorphismss

$$
Ad(u^*)_* : K_i(q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \cong K_i((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}), \qquad i = 0, 1. \tag{6.7}
$$

Let $\hat{\hat\eta}_\rho$ be the automorphism of the positive generator of $\mathbb{Z}$ for the dual action of $(\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$. Define an isomorphism

$$
\gamma_{\eta,i} = j^{\rho -1}_{\eta *} \circ Ad(u^*)^{-1}_* \circ \hat{\hat\eta}_{\rho *} \circ Ad(u^*)_* \circ j^{\rho}_{\eta *} : K_i(\mathcal{F}_{\rho,\eta}) \longrightarrow K_i(\mathcal{F}_{\rho,\eta}) \tag{6.8}
$$

such that the diagram is commutative for $i = 0, 1$:

$$
\begin{array}{ccc}
K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}) & \xrightarrow{\hat{\hat\eta}_{\rho *}} & K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}) \\
\uparrow{\scriptstyle Ad(u^*)_*} & & \uparrow{\scriptstyle Ad(u^*)_*} \\
K_i(q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) & & K_i(q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \\
\uparrow{\scriptstyle j^\rho_{\eta *}} & & \uparrow{\scriptstyle j^\rho_{\eta *}} \\
K_i(((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}) & & K_i(((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}) \\
\| & & \| \\
K_i(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\gamma_{\eta,i}} & K_i(\mathcal{F}_{\rho,\eta})
\end{array}
$$

We similarly define an endomorphism $\gamma_{\rho,i} : K_i(\mathcal{F}_{\rho,\eta}) \longrightarrow K_i(\mathcal{F}_{\rho,\eta})$.

Under the equality $((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta} = \mathcal{F}_{\rho,\eta}$, we have the following lemma which is similar to Lemma 6.2

Lemma 6.4. *The following six term exact sequence of K-theory holds:*

$$
\begin{array}{ccccc}
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\eta,0}} & K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\iota_*} & K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
{\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\exp} \\
K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xleftarrow[\iota_*]{} & K_1(\mathcal{F}_{\rho,\eta}) & \xleftarrow[\mathrm{id}-\gamma_{\eta,1}]{} & K_1(\mathcal{F}_{\rho,\eta})
\end{array}
$$

In particular, if $K_1(\mathcal{F}_{\rho,\eta}) = 0$, we have

$$K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) = \mathrm{Coker}(\mathrm{id} - \gamma_{\eta,0}) \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}), \tag{6.9}$$

$$K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) = \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}). \tag{6.10}$$

The following lemmas hold.

Lemma 6.5. *For a projection $q \in M_n((\mathcal{O}^\kappa_{\rho,\eta})^\rho)$ and a partial isometry $S \in \mathcal{O}^\kappa_{\rho,\eta}$ such that*

$$\hat\rho_z(S) = zS \quad \textit{for } z \in \mathbb{T}, \qquad q(SS^* \otimes 1_n) = (SS^* \otimes 1_n)q,$$

we have

$$\beta^{-1}_{\rho,0}([(SS^* \otimes 1_n)q]) = [(S^* \otimes 1_n)q(S \otimes 1_n)] \quad \textit{in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}).$$

Proof. As q commutes with $SS^* \otimes 1_n$, $p = (S^* \otimes 1_n)q(S \otimes 1_n)$ is a projection in $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$. Since $p \le S^*S \otimes 1_n$, By a similar argument to the proof of [20, Lemma 4.5], one sees that $\beta_{\rho,0}([p]) = [(S \otimes 1_n)p(S^* \otimes 1_n)]$ in $K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$. □

Lemma 6.6.

(i) *For a projection $q \in M_n(\mathcal{F}_{\rho,\eta})$ and a partial isometry $T \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$ such that*

$$\hat\eta_z(T) = zT \quad \textit{for } z \in \mathbb{T}, \qquad q(TT^* \otimes 1_n) = (TT^* \otimes 1_n)q,$$

we have

$$\gamma^{-1}_{\eta,0}([(TT^* \otimes 1_n)q]) = [(T^* \otimes 1_n)q(T \otimes 1_n)] \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}).$$

(ii) *For a projection $q \in M_n(\mathcal{F}_{\rho,\eta})$ and a partial isometry $S \in (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\eta}}$ such that*

$$\hat{\rho}_z(S) = zS \quad \textit{for } z \in \mathbb{T}, \qquad q(SS^* \otimes 1_n) = (SS^* \otimes 1_n)q,$$

we have

$$\gamma_{\rho,0}^{-1}([(SS^* \otimes 1_n)q]) = [(S^* \otimes 1_n)q(S \otimes 1_n)] \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}).$$

Hence we have

Lemma 6.7. *The diagram*

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta}) \\ \Big\downarrow \iota_* & & \Big\downarrow \iota_* \\ K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) & \xrightarrow{\mathrm{id}-\beta_{\rho,0}} & K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \end{array} \tag{6.11}$$

is commutative.

Proof. By [30, Proposition 3.3], the map $\iota_* : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}})$ is induced by the natural inclusion $\mathcal{F}_{\rho,\eta}(= ((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}})^{\eta}) \hookrightarrow (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}$. For an element $[q] \in K_0(\mathcal{F}_{\rho,\eta})$ one may assume that $q \in M_n(\mathcal{F}_{\rho,\eta})$ for some $n \in \mathbb{N}$ so that one has

$$\begin{aligned} \gamma_{\rho,0}^{-1}([q]) &= \sum_{\alpha \in \Sigma^{\rho}} [(S_\alpha S_\alpha^* \otimes 1_n)q] \\ &= \sum_{\alpha \in \Sigma^{\rho}} [(S_\alpha^* \otimes 1_n)q(S_\alpha \otimes 1_n)] \\ &= \sum_{\alpha \in \Sigma^{\rho}} \beta_{\rho,0}^{-1}([q(S_\alpha S_\alpha^* \otimes 1_n)]) = \beta_{\rho,0}^{-1}([q]) \end{aligned}$$

so that $\beta_{\rho,0}|_{K_0(\mathcal{F}_{\rho,\eta})} = \gamma_{\rho,0}$. □

In the rest of this section, we assume that $K_1(\mathcal{F}_{\rho,\eta}) = 0$. The following lemma is crucial in our further discussions.

Lemma 6.8. *In the six term exact sequence in Lemma 6.4 with $K_1(\mathcal{F}_{\rho,\eta}) = 0$, we have the following commutative diagrams:*

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) & \xrightarrow{\mathrm{id}-\beta_{\rho,1}} & K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
{\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow \\
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta}) \\
{\scriptstyle\mathrm{id}-\gamma_{\eta,0}}\downarrow & & {\scriptstyle\mathrm{id}-\gamma_{\eta,0}}\downarrow \\
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta}) \\
{\scriptstyle\iota_*}\downarrow & & {\scriptstyle\iota_*}\downarrow \\
K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) & \xrightarrow{\mathrm{id}-\beta_{\rho,0}} & K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\tag{6.12}
$$

Proof. It is well-known that δ-map is functorial (see [44, Theorem 7.2.5], [3, p.266 (LX)]). Hence the diagram of the upper square

$$
\begin{array}{ccc}
K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) & \xrightarrow{\mathrm{id}-\beta_{\rho,1}} & K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
{\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow \\
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta})
\end{array}
$$

is commutative.

Since $\gamma_{\rho,0} \circ \gamma_{\eta,0} = \gamma_{\eta,0} \circ \gamma_{\rho,0}$ the diagram of the middle square

$$
\begin{array}{ccc}
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta}) \\
\downarrow{\scriptstyle\mathrm{id}-\gamma_{\eta,0}} & & \downarrow{\scriptstyle\mathrm{id}-\gamma_{\eta,0}} \\
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta})
\end{array}
\tag{6.13}
$$

is commutative.

The commutativity of the lower square comes from the preceding lemma. $\square$

Lemma 6.9. *Suppose that $K_1(\mathcal{F}_{\rho,\eta}) = 0$. The six term exact sequence in Lemma 6.2 with Lemma 6.8 goes to the following commutative diagrams:*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \arrow[r, "\mathrm{id}-\beta_{\rho,1}"] \arrow[d, "\delta"] & K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \arrow[d, "\delta"] & & \\
 & K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\mathrm{id}-\gamma_{\rho,0}"] \arrow[d, "\mathrm{id}-\gamma_{\eta,0}"] & K_0(\mathcal{F}_{\rho,\eta}) \arrow[d, "\mathrm{id}-\gamma_{\eta,0}"] & & \\
 & K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\mathrm{id}-\gamma_{\rho,0}"] \arrow[d, "\iota_*"] & K_0(\mathcal{F}_{\rho,\eta}) \arrow[d, "\iota_*"] & & \\
 {} \arrow[r, "\delta"] & K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \arrow[r, "\mathrm{id}-\beta_{\rho,0}"] \arrow[d] & K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \arrow[r, "\iota_*"] \arrow[d] & K_0(\mathcal{O}^{\kappa}_{\rho,\eta}) \arrow[r, "\exp"] & {} \\
 & 0 & 0 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 {} \arrow[r, "\exp"] & K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \arrow[r, "\mathrm{id}-\beta_{\rho,1}"] \arrow[d, "\delta"] & K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \arrow[r, "\iota_*"] \arrow[d, "\delta"] & K_1(\mathcal{O}^{\kappa}_{\rho,\eta}) \arrow[r, "\delta"] & {} \\
 & K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\mathrm{id}-\gamma_{\rho,0}"] \arrow[d, "\mathrm{id}-\gamma_{\eta,0}"] & K_0(\mathcal{F}_{\rho,\eta}) \arrow[d, "\mathrm{id}-\gamma_{\eta,0}"] & & \\
 & K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\mathrm{id}-\gamma_{\rho,0}"] \arrow[d, "\iota_*"] & K_0(\mathcal{F}_{\rho,\eta}) \arrow[d, "\iota_*"] & & \\
 & K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \arrow[r, "\mathrm{id}-\beta_{\rho,0}"] \arrow[d] & K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

We will describe the K-theory groups $K_*(\mathcal{O}^{\kappa}_{\rho,\eta})$ in terms of the kernels and cokernels of the homomorphisms $\mathrm{id} - \gamma_{\rho,i}$ and $\mathrm{id} - \gamma_{\eta,i}$ on $K_0(\mathcal{F}_{\rho,\eta})$. Recall that there exist short exact sequences by Lemma 6.2:

(i)

$$
\begin{aligned}
0 &\longrightarrow \mathrm{Coker}(\mathrm{id} - \beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
&\longrightarrow K_0(\mathcal{O}^{\kappa}_{\rho,\eta}) \\
&\longrightarrow \mathrm{Ker}(\mathrm{id} - \beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) \\
&\longrightarrow 0.
\end{aligned}
$$

(ii)

$$\begin{aligned} 0 &\longrightarrow \mathrm{Coker}(\mathrm{id} - \beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\ &\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id} - \beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\ &\longrightarrow 0. \end{aligned}$$

As $\gamma_{\eta,0} \circ \gamma_{\rho,0} = \gamma_{\rho,0} \circ \gamma_{\eta,0}$ on $K_0(\mathcal{F}_{\rho,\eta})$, $\gamma_{\rho,0}$ and $\gamma_{\eta,0}$ naturally act on $\mathrm{Coker}(\mathrm{id} - \gamma_{\eta,0}) = K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$ and $\mathrm{Coker}(\mathrm{id} - \gamma_{\rho,0}) = K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})$ as endomorphisms respectively, which we denote by $\bar\gamma_{\rho,0}$ and $\bar\gamma_{\eta,0}$ respectively.

Lemma 6.10.

(i) *For $K_0(\mathcal{O}^\kappa_{\rho,\eta})$, we have*

$$\begin{aligned} &\mathrm{Coker}(\mathrm{id} - \beta_{\rho,0}) \textit{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\ \cong &\mathrm{Coker}(\mathrm{id} - \bar\gamma_{\rho,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) \\ \cong &K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Ker}(\mathrm{id} - \beta_{\rho,0}) \textit{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\ \cong &\mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \textit{ in } (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta})) \\ \cong &\mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \cap \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta}). \end{aligned}$$

(ii) *For $K_1(\mathcal{O}^\kappa_{\rho,\eta})$, we have*

$$\begin{aligned} &\mathrm{Coker}(\mathrm{id} - \beta_{\rho,1}) \textit{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\ \cong &(\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id} - \gamma_{\rho,0})(\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta})) \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Ker}(\mathrm{id} - \beta_{\rho,0}) \textit{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\ \cong &\mathrm{Ker}(\mathrm{id} - \bar\gamma_{\rho,0}) \textit{ in } (K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})). \end{aligned}$$

Proof. (i) We will first prove the assertions for the group $\mathrm{Coker}(\mathrm{id} - \beta_{\rho,0})$ in $K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$.

In the diagram (6.12), the exactness of the vertical arrows at $K_0(\mathcal{F}_{\rho,\eta})$, one sees that δ is injective and $\mathrm{Im}(\delta) = \mathrm{Ker}(\mathrm{id} - \gamma_\eta)$ so that we have

$$K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \cong \delta(K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})) \cong \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}). \tag{6.14}$$

By the commutativity in the upper square in the diagram (6.12), one has

$$\mathrm{Ker}(\mathrm{id} - \beta_{\rho,0}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \cong \mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \text{ in } (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})).$$

Since $\gamma_{\eta,0}$ commutes with $\gamma_{\rho,0}$ in $K_0(\mathcal{F}_{\rho,\eta})$, we have

$$\begin{aligned} &\mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \text{ in } (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})) \\ \cong &\mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \cap \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}). \end{aligned}$$

We will second prove the assertions for the group $\mathrm{Ker}(\mathrm{id} - \beta_{\rho,1})$ in $K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$.

In the diagram (6.12), the exactness of the vertical arrows at $K_0(\mathcal{F}_{\rho,\eta})$, one sees that ι_* is surjective so that

$$\begin{aligned}
K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \cong \iota_*(K_0(\mathcal{F}_{\rho,\eta})) \\
& \cong K_0(\mathcal{F}_{\rho,\eta})/\mathrm{Ker}(\mathrm{id}-\gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}).
\end{aligned}$$

By the commutativity in the lower square in the diagram (6.12), one has

$$\begin{aligned}
& \mathrm{Coker}(\mathrm{id}-\beta_{\rho,0}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
\cong & \mathrm{Coker}(\mathrm{id}-\bar\gamma_{\rho,0}) \text{ in } (\mathrm{Coker}(\mathrm{id}-\gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))
\end{aligned}$$

We will show that

$$\begin{aligned}
& \mathrm{Coker}(\mathrm{id}-\bar\gamma_{\rho,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) \\
\cong & K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) + (\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})).
\end{aligned}$$

Put $H_{\rho,\eta} = (\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) + (\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}))$ the subgroup of $K_0(\mathcal{F}_{\rho,\eta})$ generated by $(\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})$ and $(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$. Set the quotient maps

$$\begin{aligned}
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{q_\eta} K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) \\
& \xrightarrow{q_{(\mathrm{id}-\gamma_{\rho,0})}} \mathrm{Coker}(\mathrm{id}-\bar\gamma_{\rho,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})
\end{aligned}$$

and $\Phi = q_{(\mathrm{id}-\gamma_{\rho,0})}\circ q_\eta : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow \mathrm{Coker}(\mathrm{id}-\bar\gamma_{\rho,0})$ in $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$. As $(\mathrm{id}-\gamma_{\rho,0})$ commutes with $(\mathrm{id}-\gamma_{\eta,0})$, one has

$$(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) \subset \mathrm{Ker}(\Phi), \qquad (\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) \subset \mathrm{Ker}(\Phi).$$

Hence we have $H_{\rho,\eta} \subset \mathrm{Ker}(\Phi)$.

On the other hand, for $g \in \mathrm{Ker}(\Phi)$, we have $g \in (\mathrm{id}-\bar\gamma_{\rho,0})(K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}))$ so that $g = (\mathrm{id}-\gamma_{\rho,0})[h]$ for some $[h] \in K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$. Hence $g = (\mathrm{id}-\gamma_{\rho,0})h + (\mathrm{id}-\gamma_{\rho,0})(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$ so that $g \in H_{\rho,\eta}$. Hence we have $\mathrm{Ker}(\Phi) \subset H_{\rho,\eta}$ and $\mathrm{Ker}(\Phi) = H_{\rho,\eta}$. As

$$\begin{aligned}
& (K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id}-\bar\gamma_{\rho,0})((K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) \\
\cong & K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) + (\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})),
\end{aligned}$$

we have

$$\begin{aligned}
& \mathrm{Coker}(\mathrm{id}-\beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
\cong & K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) + (\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})).
\end{aligned}$$

(ii) The assertions are similarly shown to (i). □

Therefore we have

Theorem 6.11. *Assume that $K_1(\mathcal{F}_{\rho,\eta}) = 0$. There exist short exact sequences:*

(i)

$$\begin{aligned}
0 & \longrightarrow K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) \\
& \longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta}) \\
& \longrightarrow \mathrm{Ker}(\mathrm{id}-\gamma_{\rho,0}) \cap \mathrm{Ker}(\mathrm{id}-\gamma_{\eta,0}) \ \textit{in}\ K_0(\mathcal{F}_{\rho,\eta})) \\
& \longrightarrow 0.
\end{aligned}$$

(ii)

$$
\begin{aligned}
0 &\longrightarrow (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \ \textit{in} \ K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id} - \gamma_{\rho,0})(\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \ \textit{in} \ K_0(\mathcal{F}_{\rho,\eta}))\\
&\longrightarrow K_1(\mathcal{O}^{\kappa}_{\rho,\eta})\\
&\longrightarrow \mathrm{Ker}(id - \gamma_{\rho,0}) \ \textit{in} \ K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \bar{\gamma}_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})\\
&\longrightarrow 0.
\end{aligned}
$$

As a corollary we have

Corollary 6.12. *Suppose $K_1(\mathcal{F}_{\rho,\eta}) = 0$. We then have*

(i)

$$
\begin{aligned}
0 &\longrightarrow \mathrm{Coker}(\mathrm{id} - \bar{\gamma}_{\rho,0}) \ \textit{in} \ K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})\\
&\longrightarrow K_0(\mathcal{O}^{\kappa}_{\rho,\eta})\\
&\longrightarrow \mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \ \textit{in} \ (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \ \textit{in} \ K_0(\mathcal{F}_{\rho,\eta}))\\
&\longrightarrow 0.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
0 &\longrightarrow (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \ \textit{in} \ K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id} - \gamma_{\rho,0})(\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \ \textit{in} \ K_0(\mathcal{F}_{\rho,\eta}))\\
&\longrightarrow K_1(\mathcal{O}^{\kappa}_{\rho,\eta})\\
&\longrightarrow \mathrm{Ker}(\mathrm{id} - \bar{\gamma}_{\rho,0}) \ \textit{in} \ (K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}))\\
&\longrightarrow 0.
\end{aligned}
$$

7. K-Theory formulae

We henceforth denote the endomorphisms $\gamma_{\rho,0}, \gamma_{\eta,0}$ on $K_0(\mathcal{F}_{\rho,\eta})$ by γ_ρ, γ_η respectively.

In this section, we will prove more useful formulae for the K-groups $K_i(\mathcal{O}^{\kappa}_{\rho,\eta})$ under certain additional assumption on $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$. The assumed condition on $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is the following:
Definition. A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to *form square* if the C^*-subalgebra $C^*(\rho_\alpha(1) : \alpha \in \Sigma^\rho)$ of $\mathcal{A}$ generated by the projections $\rho_\alpha(1), \alpha \in \Sigma^\rho$ coincides with the C^*-subalgebra $C^*(\eta_a(1) : a \in \Sigma^\eta)$ of $\mathcal{A}$ generated by the projections $\eta_a(1), a \in \Sigma^\eta$.

Lemma 7.1. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square. Put for $l \in \mathbb{Z}_+$*

$$\mathcal{A}^\rho_l = C^*(\rho_\mu(1) : \mu \in B_l(\Lambda_\rho)), \qquad \mathcal{A}^\eta_l = C^*(\eta_\xi(1) : \xi \in B_l(\Lambda_\eta)).$$

Then $\mathcal{A}^\rho_l = \mathcal{A}^\eta_l$.

Proof. By assumption, we have $\mathcal{A}^\rho_1 = \mathcal{A}^\eta_1$. Hence the desired equality for $l = 1$ holds. Suppose that the equalities hold for all $l \le k$ for some $k \in \mathbb{N}$. For $\mu = \mu_1\mu_2 \cdots \mu_k\mu_{k+1} \in B_{k+1}(\Lambda_\rho)$ we have $\rho_\mu(1) = \rho_{\mu_{k+1}}(\rho_{\mu_1\mu_2\cdots\mu_k}(1))$ so that $\rho_\mu(1) \in \rho_{\mu_{k+1}}(\mathcal{A}^\rho_k)$. By the κ-commutation relation, one sees that

$$\rho_{\mu_{k+1}}(\mathcal{A}^\rho_k) \subset C^*(\eta_\xi(\rho_\alpha(1)) : \xi \in B_k(\Lambda_\eta), \alpha \in \Sigma^\rho).$$

Since $C^*(\rho_\alpha(1) : \alpha \in \Sigma^\rho) = C^*(\eta_a(1) : a \in \Sigma^\eta)$, one knows that the algebra $C^*(\eta_\xi(\rho_\alpha(1)) : \xi \in B_k(\Lambda_\eta), \alpha \in \Sigma^\rho)$ is contained in $\mathcal{A}^\eta_{k+1}$ so that $\rho_{\mu_{k+1}}(\mathcal{A}^\eta_k) \subset \mathcal{A}^\eta_{k+1}$. Therefore we have $\rho_\mu(1) \in \mathcal{A}^\eta_{k+1}$ so that $\mathcal{A}^\rho_{k+1} \subset \mathcal{A}^\eta_{k+1}$ and hence $\mathcal{A}^\rho_{k+1} = \mathcal{A}^\eta_{k+1}$. □

Therefore we have

Lemma 7.2. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square. Put for $j, k \in \mathbb{Z}_+$*

$$\begin{aligned}\mathcal{A}_{j,k} &= C^*(\rho_\mu(\eta_\zeta(1)) : \mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta)) \\ &(= C^*(\eta_\xi(\rho_\nu(1)) : \xi \in B_k(\Lambda_\eta), \nu \in B_j(\Lambda_\rho))).\end{aligned}$$

Then $\mathcal{A}_{j,k}$ is commutative and of finite dimensional such that

$$\mathcal{A}_{j,k} = \mathcal{A}^\rho_{j+k} (= \mathcal{A}^\eta_{j+k}).$$

Hence $\mathcal{A}_{j,k} = \mathcal{A}_{j',k'}$ if $j + k = j' + k'$.

Proof. Since $\eta_\zeta(1) \in Z_{\mathcal{A}}$ and $\rho_\mu(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$, the algebra $\mathcal{A}_{j,k}$ belongs to the center $Z_{\mathcal{A}}$ of $\mathcal{A}$. By the preceding lemma, we have

$$\mathcal{A}_{j,k} = C^*(\rho_\mu(\rho_\nu(1)) : \mu \in B_j(\Lambda_\rho), \nu \in B_k(\Lambda_\rho)) = \mathcal{A}^\rho_{j+k}.$$

□

For $j, k \in \mathbb{Z}_+$, put $l = j + k$. We denote by $\mathcal{A}_l$ the commutative finite dimensional algebra $\mathcal{A}_{j,k}$. Put $m(l) = \dim\mathcal{A}_l$. Take the finite sequence of minimal projections $E^l_i, i = 1, 2, \dots, m(l)$ in $\mathcal{A}_l$ such that $\sum_{i=1}^{m(l)} E^l_i = 1$. Hence we have $\mathcal{A}_l = \sum_{i=1}^{m(l)} \mathbb{C}E^l_i$. Since $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$, there exists $A^\rho_{l,l+1}(i, \alpha, n)$, which takes 0 or 1, such that

$$\rho_\alpha(E^l_i) = \sum_{n=1}^{m(l+1)} A^\rho_{l,l+1}(i, \alpha, n)E^{l+1}_n, \qquad \alpha \in \Sigma^\rho,\ i = 1, \dots, m(l).$$

Similarly, there exists $A^\eta_{l,l+1}(i, a, n)$, which takes 0 or 1, such that

$$\eta_a(E^l_i) = \sum_{n=1}^{m(l+1)} A^\eta_{l,l+1}(i, a, n)E^{l+1}_n, \qquad a \in \Sigma^\eta,\ i = 1, \dots, m(l).$$

Let $N_{j,k}(i)$ be the cardinal number of the set

$$\{(\mu, \zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta) \mid \rho_\mu(\eta_\zeta(1)) \ge E^l_i\}.$$

Set for $i = 1, \dots, m(l)$

$$\begin{aligned}\mathcal{F}_{j,k}(i) &= C^*(S_\mu T_\zeta E^l_i x E^l_i T^*_\xi S^*_\nu \mid \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}) \\ &= C^*(T_\zeta S_\mu E^l_i x E^l_i S^*_\nu T^*_\xi \mid \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}).\end{aligned}$$

Since E^l_i is a central projection in $\mathcal{A}$, we have

Lemma 7.3. (i) *$\mathcal{F}_{j,k}(i)$ is isomorphic to the matrix algebra $M_{N_{j,k}(i)}(E^l_i\mathcal{A}E^l_i)(= M_{N_{j,k}(i)}(\mathbb{C}) \otimes E^l_i\mathcal{A}E^l_i)$ over $E^l_i\mathcal{A}E^l_i$.*
(ii) *$\mathcal{F}_{j,k} = \mathcal{F}_{j,k}(1) \oplus \cdots \oplus \mathcal{F}_{j,k}(m(l))$.*

Proof. (i) For $(\mu, \zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta)$ with $S_\mu T_\zeta E^l_i \ne 0$, one has $\eta_\zeta(\rho_\mu(1))E^l_i \ne 0$ so that $\eta_\zeta(\rho_\mu(1)) \ge E^l_i$. Hence $(S_\mu T_\zeta E^l_i)^* S_\mu T_\zeta E^l_i = E_i$. One sees that the set

$$\{S_\mu T_\zeta E^l_i \mid (\mu, \zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta); S_\mu T_\zeta E^l_i \ne 0\}$$

consist of isometries which give rise to matrix units of $\mathcal{F}_{j,k}(i)$ such that $\mathcal{F}_{j,k}(i)$ is isomorphic to $M_{N_{j,k}(i)}(E^l_i\mathcal{A}E^l_i)$.

(ii) Since $\mathcal{A} = E^l_1\mathcal{A}E^l_1 \oplus \cdots \oplus E^l_{m(l)}\mathcal{A}E^l_{m(l)}$ the assertion is easy. □

Define $\lambda_{\rho *}, \lambda_{\eta *} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ by setting

$$\lambda_{\rho *}([p]) = \sum_{\alpha \in \Sigma^\rho} [\rho_\alpha \otimes 1_n(p)], \qquad \lambda_{\eta *}([p]) = \sum_{a \in \Sigma^\eta} [\eta_a \otimes 1_n(p)]$$

for a projection $p \in M_n(\mathcal{A})$ for some $n \in \mathbb{N}$.

Lemma 7.4. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square. There exists an isomorphism*

$$\Phi_{j,k} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{A})$$

such that the following diagrams are commutative:

(i)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) \\ {\scriptstyle \Phi_{j,k}}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi_{j+1,k}} \\ K_0(\mathcal{A}) & \xrightarrow{\lambda_{\rho *}} & K_0(\mathcal{A}) \end{array}$$

(ii)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{j,k+1}) \\ {\scriptstyle \Phi_{j,k}}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi_{j,k+1}} \\ K_0(\mathcal{A}) & \xrightarrow{\lambda_{\eta *}} & K_0(\mathcal{A}) \end{array}$$

Proof. Put for $i = 1, 2, \cdots m(l)$

$$P_i = \sum_{\mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta)} S_\mu T_\zeta E_i^l T_\zeta^* S_\mu^*$$

Then P_i is a projection which belongs to the center of $\mathcal{F}_{j,k}$ such that $\sum_{i=1}^{m(l)} P_i = 1$. For $X \in \mathcal{F}_{j,k}$, one has $P_i X P_i \in \mathcal{F}_{j,k}(i)$ such that

$$X = \sum_{i=1}^{m(l)} P_i X P_i \in \bigoplus_{i=1}^{m(l)} \mathcal{F}_{j,k}(i).$$

Define an isomorphism

$$\varphi_{j,k} : X \in \mathcal{F}_{j,k} \longrightarrow \sum_{i=1}^{m(l)} P_i X P_i \in \bigoplus_{i=1}^{m(l)} \mathcal{F}_{j,k}(i)$$

which induces an isomorphism on their K-groups

$$\varphi_{j,k*} : K_0(\mathcal{F}_{j,k}) \longrightarrow \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)).$$

Take and fix $\nu(i), \mu(i) \in B_j(\Lambda_\rho)$ and $\zeta(i), \xi(i) \in B_k(\Lambda_\eta)$ such that

$$T_{\xi(i)} S_{\nu(i)} = S_{\mu(i)} T_{\zeta(i)} \quad \text{ and } \quad T_{\xi(i)} S_{\nu(i)} E_i^l \neq 0.$$

Hence $S_{\nu(i)}^* T_{\xi(i)}^* T_{\xi(i)} S_{\nu(i)} \geq E_i^l$. Since $\mathcal{F}_{j,k}(i)$ is isomorphic to $M_{N_{j,k(i)}}(\mathbb{C}) \otimes E_i^l \mathcal{A} E_i^l$, the embedding

$$\iota_{j,k}(i) : x \in E_i^l \mathcal{A} E_i^l \longrightarrow T_{\xi(i)} S_{\nu(i)} x S_{\nu(i)}^* T_{\xi(i)}^* \in \mathcal{F}_{j,k}(i)$$

induces an isomorphism on their K-groups

$$\iota_{j,k}(i)_* : K_0(E_i^l \mathcal{A} E_i^l) \longrightarrow K_0(\mathcal{F}_{j,k}(i)).$$

Put

$$\psi_{j,k} = \oplus_{i=1}^{m(l)} \iota_{j,k}(i) : \oplus_{i=1}^{m(l)} E_i^l \mathcal{A} E_i^l \longrightarrow \oplus_{i=1}^{m(l)} \mathcal{F}_{j,k}(i)$$

and hence

$$\psi_{j,k*} = \bigoplus_{i=1}^{m(l)} \iota_{i*} : \bigoplus_{i=1}^{m(l)} K_0(E_i^l \mathcal{A} E_i^l) \longrightarrow \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)).$$

Hence we have isomorphisms

$$K_0(\mathcal{F}_{j,k}) \xrightarrow{\varphi_{j,k*}} \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)) \xrightarrow{{\psi_{j,k*}}^{-1}} \bigoplus_{i=1}^{m(l)} K_0(E_i^l \mathcal{A} E_i^l).$$

Since $K_0(\mathcal{A}) = \bigoplus_{i=1}^{m(l)} K_0(E_i^l \mathcal{A} E_i^l)$, we have an isomorphism

$$\Phi_{j,k} = {\psi_{j,k*}}^{-1} \circ \varphi_{j,k*} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{A}).$$

(i) It suffices to show the following diagram

$$\begin{array}{ccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) \\
{\scriptstyle \varphi_{j,k*}} \downarrow & & {\scriptstyle \varphi_{j+1,k*}} \downarrow \\
\bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)) & & \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j+1,k}(i)) \\
{\scriptstyle \psi_{j,k*}} \uparrow & & {\scriptstyle \psi_{j+1,k*}} \uparrow \\
K_0(\mathcal{A}) & \xrightarrow{\lambda_{\rho*}} & K_0(\mathcal{A})
\end{array}$$

is commutative. For $a = \sum_{i=1}^{m(l)} E_i^l a E_i^l \in \mathcal{A}$, we have

$$\psi_{j,k}(a) = \sum_{i=1}^{m(l)} T_{\xi(i)} S_{\nu(i)} E_i^l a E_i^l S_{\nu(i)}^* T_{\xi(i)}^* = \sum_{i=1}^{m(l)} S_{\mu(i)} T_{\zeta(i)} E_i^l a E_i^l T_{\zeta(i)}^* S_{\mu(i)}^*.$$

Since $P_i T_{\xi(i)} S_{\nu(i)} E_i^l a E_i^l S_{\nu(i)}^* T_{\xi(i)}^* P_i = T_{\xi(i)} S_{\nu(i)} E_i^l a E_i^l S_{\nu(i)}^* T_{\xi(i)}^*$, we have

$$\varphi_{j,k}^{-1} \circ \psi_{j,k}(a) = \sum_{i=1}^{m(l)} T_{\xi(i)} S_{\nu(i)} E_i^l a E_i^l S_{\nu(i)}^* T_{\xi(i)}^*$$

so that

$$\iota_{+1,*} \circ \varphi_{j,k}^{-1} \circ \psi_{j,k}(a) = \sum_{\alpha \in \Sigma} \sum_{i=1}^{m(l)} T_{\xi(i)} S_{\nu(i)\alpha} \rho_\alpha(E_i^l a E_i^l) S_{\nu(i)\alpha}^* T_{\xi(i)}^*.$$

Since

$$S_{\nu(i)\alpha} \rho_\alpha(E_i^l a E_i^l) S_{\nu(i)\alpha}^* = \sum_{n=1}^{m(l+1)} A_{l,l+1}^{\rho}(i, \alpha, n) S_{\nu(i)\alpha} E_n^{l+1} \rho_\alpha(a) E_n^{l+1} S_{\nu(i)\alpha}^*$$

and $A^{\rho}_{l,l+1}(i,\alpha,n)S_{\nu(i)\alpha}E^{l+1}_n = S_{\nu(i)\alpha}E^{l+1}_n$, we have

$$\sum_{\alpha\in\Sigma^{\rho}} S_{\nu(i)\alpha}\rho_{\alpha}(E^l_i a E^l_i)S^*_{\nu(i)\alpha} = \sum_{n=1}^{m(l+1)}\sum_{\alpha\in\Sigma^{\rho}} S_{\nu(i)\alpha}E^{l+1}_n\rho_{\alpha}(a)E^{l+1}_n S^*_{\nu(i)\alpha}$$

so that

$$\iota_{+1,*}\circ\varphi^{-1}_{j,k}\circ\psi_{j,k}(a) = \sum_{\alpha\in\Sigma}\sum_{i=1}^{m(l)}\sum_{n=1}^{m(l+1)} T_{\xi(i)}S_{\nu(i)\alpha}E^{l+1}_n\rho_{\alpha}(a)E^{l+1}_n S^*_{\nu(i)\alpha}T^*_{\xi(i)}.$$

On the other hand,

$$\begin{aligned}\psi_{j,k}(\lambda_{\rho}(a)) &= \psi_{j,k}(\sum_{\alpha\in\Sigma^{\rho}}\rho_{\alpha}(a))\\ &= \psi_{j,k}(\sum_{n=1}^{m(l+1)}\sum_{\alpha\in\Sigma^{\rho}} E^{l+1}_n\rho_{\alpha}(a))E^{l+1}_n\\ &= \sum_{\alpha\in\Sigma}\sum_{i=1}^{m(l)}\sum_{n=1}^{m(l+1)} T_{\xi(i)}S_{\nu(i)\alpha}E^{l+1}_n\rho_{\alpha}(a)E^{l+1}_n S^*_{\nu(i)\alpha}T^*_{\xi(i)}.\end{aligned}$$

Therefore we have

$$\iota_*\circ\varphi^{-1}_{j,k}\circ\psi_{j,k}(a) = \psi_{j,k}(\lambda_{\rho}(a)).$$

(ii) is symmetric to (i). □

Define the abelian groups of inductive limits:

$$G_{\rho} = \lim\{\lambda_{\rho} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}, \qquad G_{\eta} = \lim\{\lambda_{\eta} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}.$$

Put for $j,k\in\mathbb{Z}_+$ the subalgebras of $\mathcal{F}_{\rho,\eta}$

$$\begin{aligned}\mathcal{F}_{\rho,k} &= C^*(T_{\zeta}S_{\mu}xS^*_{\nu}T^*_{\xi} \mid \mu,\nu\in B_*(\Lambda_{\rho}), |\mu|=|\nu|, \zeta,\xi\in B_k(\Lambda_{\eta}), x\in\mathcal{A})\\ &= C^*(T_{\zeta}yT^*_{\xi} \mid \zeta,\xi\in B_k(\Lambda_{\eta}), y\in\mathcal{F}_{\rho})\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_{j,\eta} &= C^*(S_{\mu}T_{\zeta}xT^*_{\xi}S^*_{\nu} \mid \mu,\nu\in B_j(\Lambda_{\rho}), \zeta,\xi\in B_*(\Lambda_{\eta}), |\zeta|=|\xi|, x\in\mathcal{A})\\ &= C^*(S_{\mu}yS^*_{\nu} \mid \mu,\nu\in B_j(\Lambda_{\rho}), y\in\mathcal{F}_{\eta}).\end{aligned}$$

Lemma 7.5. *For $j,k\in\mathbb{Z}_+$, there exist isomorphisms*

$$\Phi_{\rho,k} : K_0(\mathcal{F}_{\rho,k}) \longrightarrow G_{\rho}, \qquad \Phi_{j,\eta} : K_0(\mathcal{F}_{j,\eta}) \longrightarrow G_{\eta}$$

such that the following diagrams are commutative:

(i)

$$\begin{array}{ccccccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) & \xrightarrow{\iota_{+1,*}} & \cdots & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{\rho,k})\\ {\scriptstyle\Phi_{j,k}}\downarrow & & {\scriptstyle\Phi_{j+1,k}}\downarrow & & & & {\scriptstyle\Phi_{\rho,k}}\downarrow\\ K_0(\mathcal{A}) & \xrightarrow{\lambda_{\rho*}} & K_0(\mathcal{A}) & \xrightarrow{\lambda_{\rho*}} & \cdots & \xrightarrow{\lambda_{\rho*}} & G_{\rho}\end{array}$$

(ii)

$$\begin{tikzcd}
K_0(\mathcal{F}_{j,k}) \arrow[r, "\iota_{*,+1}"] \arrow[d, "\Phi_{j,k}"'] & K_0(\mathcal{F}_{j,k+1}) \arrow[r, "\iota_{*,+1}"] \arrow[d, "\Phi_{j,k+1}"'] & \cdots \arrow[r, "\iota_{*,+1}"] & K_0(\mathcal{F}_{j,\eta}) \arrow[d, "\Phi_{j,\eta}"'] \\
K_0(\mathcal{A}) \arrow[r, "\lambda_{\eta *}"] & K_0(\mathcal{A}) \arrow[r, "\lambda_{\eta *}"] & \cdots \arrow[r, "\lambda_{\eta *}"] & G_\eta
\end{tikzcd}$$

Lemma 7.6. *If $\xi = \xi_1 \cdots \xi_k \in B_k(\Lambda_\eta), \nu = \nu_1 \cdots \nu_j \in B_j(\Lambda_\rho)$ and $i = 1, \dots, m(l)$ satisfy the condition $\rho_\nu(\eta_\xi(1)) \geq E_i^l$ where $l = j + k$, then $T_{\xi_1}^* T_\xi S_\nu E_i^l = T_{\bar{\xi}} S_\nu E_i^l$ where $\bar{\xi} = \xi_2 \cdots \xi_k$.*

Proof. Since $T_{\xi_1}^* T_\xi = T_{\xi_1}^* T_{\xi_1} T_{\bar{\xi}} T_{\bar{\xi}}^* T_{\bar{\xi}} = T_{\bar{\xi}} T_{\bar{\xi}}^* T_{\xi_1}^* T_{\xi_1} T_{\bar{\xi}} = T_{\bar{\xi}} T_{\bar{\xi}}^* T_\xi$, we have

$$T_{\xi_1}^* T_\xi S_\nu E_i^l = T_{\bar{\xi}} S_\nu S_\nu^* T_{\bar{\xi}}^* T_\xi S_\nu E_i^l = T_{\bar{\xi}} S_\nu \rho_\nu(\eta_\xi(1)) E_i^l = T_{\bar{\xi}} S_\nu E_i^l.$$

□

Lemma 7.7. *For k, j we have*

(i) *The restriction of γ_η^{-1} to $K_0(\mathcal{F}_{j,k})$ makes the following diagram commutative:*

$$\begin{tikzcd}
K_0(\mathcal{F}_{j,k}) \arrow[r, "\gamma_\eta^{-1}"] \arrow[d, "\Phi_{j,k}"'] & K_0(\mathcal{F}_{j,k-1}) \arrow[r, "\iota_{*,+1}"] & K_0(\mathcal{F}_{j,k}) \arrow[d, "\Phi_{j,k}"'] \\
K_0(\mathcal{A}) \arrow[rr, "\lambda_{\eta *}"] & & K_0(\mathcal{A}).
\end{tikzcd}$$

(ii) *The restriction of γ_ρ^{-1} to $K_0(\mathcal{F}_{j,k})$ makes the following diagram commutative:*

$$\begin{tikzcd}
K_0(\mathcal{F}_{j,k}) \arrow[r, "\gamma_\rho^{-1}"] \arrow[d, "\Phi_{j,k}"'] & K_0(\mathcal{F}_{j-1,k}) \arrow[r, "\iota_{+1,*}"] & K_0(\mathcal{F}_{j,k}) \arrow[d, "\Phi_{j,k}"'] \\
K_0(\mathcal{A}) \arrow[rr, "\lambda_{\rho *}"] & & K_0(\mathcal{A}).
\end{tikzcd}$$

Proof. (i) Put $l = j + k$. Take a projection $p \in M_n(\mathcal{A})$ for some $n \in \mathbb{N}$. Since $\mathcal{A} \otimes M_n(\mathbb{C}) = \sum_{i=1}^{m(l)} (E_i^l \otimes 1)(\mathcal{A} \otimes M_n)(E_i^l \otimes 1)$, by putting $p_i^l = (E_i^l \otimes 1) p (E_i^l \otimes 1) \in (E_i^l \otimes 1)(\mathcal{A} \otimes M_n)(E_i^l \otimes 1) = M_n(E_i^l \mathcal{A} E_i^l)$, we have $p = \sum_{i=1}^{m(l)} p_i^l$. Take $\xi(i) = \xi_1(i) \cdots \xi_k(i) \in B_k(\Lambda_\eta), \nu(i) = \nu_1(i) \cdots \nu_j(i) \in B_j(\Lambda_\rho)$, such that $\rho_{\nu(i)}(\eta_{\xi(i)}(1)) \geq E_i^l$ and put $\bar{\xi}(i) = \xi_2(i) \cdots \xi_k(i)$ so that $\xi(i) = \xi_1(i) \bar{\xi}(i)$. Since

$$\psi_{j,k*}([p]) = \sum_{i=1}^{m(l)} \oplus [(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \in \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)).$$

As

$$(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n) \leq T_{\xi_1(i)} T_{\xi_1(i)}^* \otimes 1_n,$$

by the preceding lemma we have

$$T_{\xi_1(i)}^* T_{\xi(i)} S_{\nu(i)} E_i^l = T_{\bar{\xi}(i)} S_{\nu(i)} E_i^l$$

so that

$$\gamma_\eta^{-1}([(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] = [(T_{\bar{\xi}(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\bar{\xi}(i)}^* \otimes 1_n)].$$

Hence $K_0(\mathcal{F}_{j,k})$ goes to $K_0(\mathcal{F}_{j,k-1})$ by the homomorphism γ_η^{-1}. Take $\mu(i) \in B_j(\Lambda_\rho), \bar{\zeta}(i) \in B_{k-1}(\Lambda_\eta)$ such that $T_{\bar{\xi}(i)} S_{\nu(i)} = S_{\mu(i)} T_{\bar{\zeta}(i)}$ for $i = 1, \dots, m(l)$. The element

$$
\begin{aligned}
&\sum_{i=1}^{m(l)} [(T_{\bar{\xi}(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S^*_{\nu(i)} T^*_{\bar{\xi}(i)} \otimes 1_n)] \\
=&\sum_{i=1}^{m(l)} [(S_{\mu(i)} T_{\bar{\zeta}(i)} \otimes 1_n) p_i^l (T^*_{\bar{\zeta}(i)} S^*_{\mu(i)} \otimes 1_n)] \in K_0(\mathcal{F}_{j,k-1})
\end{aligned}
$$

goes to

$$
\sum_{i=1}^{m(l)} \sum_{a \in \Sigma^\eta} [(S_{\mu(i)} T_{\bar{\zeta}(i)a} \otimes 1_n)(T_a^* \otimes 1_n) p_i^l (T_a \otimes 1_n)(T^*_{\bar{\zeta}(i)a} S^*_{\mu(i)} \otimes 1_n)] \in K_0(\mathcal{F}_{j,k})
$$

by $\iota_{*,+1}$. The element is expressed as

$$
\sum_{h=1}^{m(l)} \oplus \sum_{i=1}^{m(l)} \sum_{a \in \Sigma^\eta} [(S_{\mu(i)} T_{\bar{\zeta}(i)a} \otimes 1_n) E_h^l (T_a^* \otimes 1_n) p_i^l (T_a \otimes 1_n) E_h^l (T^*_{\bar{\zeta}(i)a} S^*_{\mu(i)} \otimes 1_n)] \tag{7.1}
$$

in $\bigoplus_{h=1}^{m(l)} K_0(\mathcal{F}_{j,k}(h))$.

On the other hand,

$$
\lambda_{\eta *}([p]) = \sum_{a \in \Sigma^\eta} [(T_a^* \otimes 1_n) p (T_a \otimes 1_n)] \in K_0(\mathcal{A}).
$$

The element

$$
\sum_{a \in \Sigma^\eta} [(T_a^* \otimes 1_n) p (T_a \otimes 1_n)] = \sum_{h=1}^{m(l)} \oplus \sum_{a \in \Sigma^\eta} [E_h^l (T_a^* \otimes 1_n) p (T_a \otimes 1_n) E_h^l] \in \bigoplus_{h=1}^{m(l)} K_0(E_h^l \mathcal{A} E_h^l)
$$

is expressed as

$$
\begin{aligned}
&\sum_{h=1}^{m(l)} \oplus \sum_{a \in \Sigma^\eta} [(T_{\xi(h)} S_{\nu(h)} E_h^l \otimes 1_n)(T_a^* \otimes 1_n) p (T_a \otimes 1_n)(E_h^l S^*_{\nu(h)} T^*_{\xi(h)} \otimes 1_n)] \\
=&\sum_{h=1}^{m(l)} \oplus \sum_{a \in \Sigma^\eta} \sum_{i=1}^{m(l)} [(T_{\xi(h)} S_{\nu(h)} E_h^l \otimes 1_n)(T_a^* \otimes 1_n) p_i^l (T_a \otimes 1_n)(E_h^l S^*_{\nu(h)} T^*_{\xi(h)} \otimes 1_n)]
\end{aligned}
$$

in $\bigoplus_{h=1}^{m(l)} K_0(\mathcal{F}_{j,k}(h))$. Take $\mu'(h) \in B_j(\Lambda_\rho), \zeta'(h) \in B_k(\Lambda_\eta)$ such that $T_{\xi(h)} S_{\nu(h)} = S_{\mu'(h)} T_{\zeta'(h)}$ so that the above element is

$$
\sum_{h=1}^{m(l)} \oplus \sum_{i=1}^{m(l)} \sum_{a \in \Sigma^\eta} [(S_{\mu'(h)} T_{\zeta'(h)} E_h^l \otimes 1_n)(T_a^* \otimes 1_n) p_i^l (T_a \otimes 1_n)(E_h^l T^*_{\zeta'(h)} S^*_{\nu'(h)} \otimes 1_n)] \tag{7.2}
$$

in $\bigoplus_{h=1}^{m(l)} K_0(\mathcal{F}_{j,k}(h))$. Since for $h, i = 1, \dots, m(l), a \in \Sigma^\eta$, the classes of the K-groups coincide such as

$$
\begin{aligned}
&[(S_{\mu(i)} T_{\bar{\zeta}(i)a} \otimes 1_n) E_h^l (T_a^* \otimes 1_n) p_i^l (T_a \otimes 1_n) E_h^l (T^*_{\bar{\zeta}(i)a} S^*_{\mu(i)} \otimes 1_n)] \\
=&[(S_{\mu'(h)} T_{\zeta'(h)} E_h^l \otimes 1_n)(T_a^* \otimes 1_n) p_i^l (T_a \otimes 1_n)(E_h^l T^*_{\zeta'(h)} S^*_{\nu'(h)} \otimes 1_n)] \in K_0(\mathcal{F}_{j,k}(h))
\end{aligned}
$$

we have the element of (7.1) is equal to the element of (7.2) in $K_0(\mathcal{F}_{j,k})$. Therefore (i) holds.

(ii) is similar to (i). $\square$

The following lemma is direct.

Lemma 7.8. *For k, j the following diagrams are commutative.*

(i)

$$
\begin{array}{ccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{j,k-1}) \\
{\scriptstyle \iota_{+1,*}}\downarrow & & {\scriptstyle \iota_{+1,*}}\downarrow \\
K_0(\mathcal{F}_{j+1,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{j+1,k-1}).
\end{array}
$$

Hence γ_η^{-1} yields a homomorphism from $K_0(\mathcal{F}_{\rho,k}) = \lim_j\{\iota_{+1,*} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j+1,k})\}$ to $K_0(\mathcal{F}_{\rho,k-1}) = \lim_j\{\iota_{+1,*} : K_0(\mathcal{F}_{j,k-1}) \longrightarrow K_0(\mathcal{F}_{j+1,k-1})\}$.

(ii)

$$
\begin{array}{ccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,k}) \\
{\scriptstyle \iota_{*,+1}}\downarrow & & {\scriptstyle \iota_{*,+1}}\downarrow \\
K_0(\mathcal{F}_{j,k+1}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,k+1})
\end{array}
$$

Hence γ_ρ^{-1} yields a homomorphism from $K_0(\mathcal{F}_{j,\eta}) = \lim_k\{\iota_{*,+1} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j,k+1})\}$ to $K_0(\mathcal{F}_{j-1,\eta}) = \lim_k\{\iota_{*,+1} : K_0(\mathcal{F}_{j-1,k}) \longrightarrow K_0(\mathcal{F}_{j-1,k+1})\}$.

Lemma 7.9. *For k, j the following diagrams are commutative.*

(i)

$$
\begin{array}{ccc}
K_0(\mathcal{F}_{\rho,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,k-1}) \\
{\scriptstyle \iota_{*,+1}}\downarrow & & {\scriptstyle \iota_{*,+1}}\downarrow \\
K_0(\mathcal{F}_{\rho,k+1}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,k})
\end{array}
$$

(ii)

$$
\begin{array}{ccc}
K_0(\mathcal{F}_{j,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,\eta}) \\
{\scriptstyle \iota_{+1,*}}\downarrow & & {\scriptstyle \iota_{+1,*}}\downarrow \\
K_0(\mathcal{F}_{j+1,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j,\eta})
\end{array}
$$

Proof. (i) As in the proof of Lemma 7.8, one may take an element of $K_0(\mathcal{F}_{\rho,k})$ as in the following form:

$$
\sum_{i=1}^{m(l)} \oplus[(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \in \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i))
$$

for some projection $p \in M_n(\mathcal{A})$ and j, l with $l = j+k$, where $p_i^l = (E_i^l \otimes 1) p (E_i^l \otimes 1) \in (E_i^l \otimes 1)(\mathcal{A} \otimes M_n)(E_i^l \otimes 1) = M_n(E_i^l \mathcal{A} E_i^l)$. Let $\xi(i) = \xi_1(i)\bar{\xi}(i)$ with $\xi_1(i) \in \Sigma^\eta$, $\bar{\xi}(i) \in B_{k-1}(\Lambda_\eta)$. One may assume that $T_{\xi(i)} S_{\nu(i)} \neq 0$ so that $T_{\bar{\xi}(i)} S_{\nu(i)} = S_{\nu(i)'} T_{\bar{\xi}(i)'}$ for

some $\nu(i)' \in B_j(\Lambda_\rho), \bar{\xi}(i)' \in B_{k-1}(\Lambda_\eta)$. As in the proof of Lemma 7.8, one has

$$
\begin{aligned}
&\gamma_\eta^{-1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \\
=&[(T_{\bar{\xi}(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\bar{\xi}(i)}^* \otimes 1_n)] \\
=&[(S_{\nu(i)'}T_{\bar{\xi}(i)'} \otimes 1_n)p_i^l(S_{\nu(i)'}^* T_{\bar{\xi}(i)'}^* \otimes 1_n)]
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\iota_{*,+1} \circ \gamma_\eta^{-1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \\
=&\iota_{*,+1}([S_{\nu(i)'}T_{\bar{\xi}(i)'} \otimes 1_n)p_i^l(T_{\bar{\xi}(i)'}^* S_{\nu(i)'}^* \otimes 1_n]) \\
=&\sum_{b \in \Sigma^\eta} [(S_{\nu(i)'}T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* \otimes 1_n)]
\end{aligned}
$$

On the other hand, we have $T_{\xi(i)}S_{\nu(i)} = T_{\xi(i)_1}T_{\bar{\xi}(i)}S_{\nu(i)} = T_{\xi(i)_1}S_{\nu(i)'}T_{\bar{\xi}(i)'}$ so that

$$
\begin{aligned}
&\iota_{*,+1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \\
=&\sum_{b \in \Sigma^\eta} [(T_{\xi(i)_1}S_{\nu(i)'}T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* T_{\xi(i)_1}^* \otimes 1_n)]
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\gamma_\eta^{-1} \circ \iota_{*,+1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \\
=&\sum_{b \in \Sigma^\eta} \gamma_\eta^{-1}([(T_{\xi(i)_1}S_{\nu(i)'}T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* T_{\xi(i)_1}^* \otimes 1_n)]) \\
=&\sum_{b \in \Sigma^\eta} [(S_{\nu(i)'}T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* \otimes 1_n)].
\end{aligned}
$$

(ii) The assertion is completely symmetric to the above proof. □

Lemma 7.10. *For k, j the following diagrams are commutative.*

(i)

$$
\begin{array}{ccccc}
K_0(\mathcal{F}_{\rho,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,k-1}) & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{\rho,k}) \\
{\scriptstyle\Phi_{\rho,k}}\downarrow & & & & {\scriptstyle\Phi_{\rho,k}}\downarrow \\
G_\rho & & \xrightarrow{\lambda_{\eta*}} & & G_\rho.
\end{array}
$$

(ii)

$$
\begin{array}{ccccc}
K_0(\mathcal{F}_{j,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,\eta}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j,\eta}) \\
{\scriptstyle\Phi_{j,\eta}}\downarrow & & & & {\scriptstyle\Phi_{j,\eta}}\downarrow \\
G_\eta & & \xrightarrow{\lambda_{\rho*}} & & G_\eta.
\end{array}
$$

Proof. (i) As in the proof of Lemma 7.8 and Lemma 7.10 one may take an element of $K_0(\mathcal{F}_{\rho,k})$ as in the following form:

$$
\sum_{i=1}^{m(l)} \oplus [(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \in \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i))
$$

for some projection $p \in M_n(\mathcal{A})$ and j, l with $l = j+k$, where $p_i^l = (E_i^l \otimes 1)p(E_i^l \otimes 1) \in (E_i^l \otimes 1)(\mathcal{A} \otimes M_n)(E_i^l \otimes 1) = M_n(E_i^l \mathcal{A} E_i^l)$. Keep the notations as in the proof of Lemma 8.10, we have

$$
\begin{aligned}
&\iota_{*,+1} \circ \gamma_\eta^{-1}([(T_{\xi(i)} S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)]) \\
=&\sum_{b \in \Sigma^\eta} [(S_{\nu(i)'} T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* \otimes 1_n)]
\end{aligned}
$$

so that

$$
\begin{aligned}
&\Phi_{\rho,k} \circ \iota_{*,+1} \circ \gamma_\eta^{-1}([(T_{\xi(i)} S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \\
=&\sum_{b \in \Sigma^\eta} \Phi_{\rho,k} \circ ([S_{\nu(i)'} T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* \otimes 1_n])) \\
=&\sum_{b \in \Sigma^\eta} [(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)) \\
=&\sum_{b \in \Sigma^\eta} [(\eta_b \otimes 1_n)(p_i^l)] \\
=&\lambda_{\eta *}([p_i^l]) \\
=&\lambda_{\eta *} \circ \Phi_{\rho,k}([(T_{\xi(i)} S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)]).
\end{aligned}
$$

Therefore we have $\Phi_{\rho,k} \circ \iota_* \circ \gamma_\eta^{-1} = \lambda_{\eta *} \circ \Phi_{\rho,k}$.

(ii) The assertion is completely symmetric to the above proof. $\square$

Put

$$
\begin{aligned}
G_{\rho,k} &= K_0(\mathcal{F}_{\rho,k})(\cong G_\rho = \lim\{\lambda_{\rho,*} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}), \\
G_{j,\eta} &= K_0(\mathcal{F}_{j,\eta})(\cong G_\eta = \lim\{\lambda_{\eta,*} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}).
\end{aligned}
$$

Lemma 7.11. *The following diagrams are commutative:*

(i)

$$
\begin{array}{ccc}
K_0(\mathcal{F}_{\rho,k}) & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{\rho,k+1}) \\
\| & & \| \\
G_{\rho,k} & \xrightarrow{\lambda_{\eta *}} & G_{\rho,k+1}
\end{array}
$$

(ii)

$$
\begin{array}{ccc}
K_0(\mathcal{F}_{j,\eta}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,\eta}) \\
\| & & \| \\
G_{j,\eta} & \xrightarrow{\lambda_{\rho *}} & G_{j+1,\eta}
\end{array}
$$

Since

$$
\begin{aligned}
K_0(\mathcal{F}_{\rho,\eta}) &= \lim_k \{\iota_{*,+1} : K_0(\mathcal{F}_{\rho,k}) \longrightarrow K_0(\mathcal{F}_{\rho,k+1})\} \\
&= \lim_j \{\iota_{+1,*} : K_0(\mathcal{F}_{j,\eta}) \longrightarrow K_0(\mathcal{F}_{j,\eta})\},
\end{aligned}
$$

by putting $G_{\rho,\eta} = K_0(\mathcal{F}_{\rho,\eta})$, one has

$$\begin{aligned} G_{\rho,\eta} &= \lim_{k}\{\lambda_{\eta*} : G_{\rho,k} \longrightarrow G_{\rho,k+1}\} \\ &= \lim_{j}\{\lambda_{\rho*} : G_{j,\eta} \longrightarrow G_{j,\eta}\}. \end{aligned}$$

Define endomorphisms

$$\begin{aligned} \sigma_\eta \text{ on } G_{\rho,\eta} &= \lim_{k}\{\lambda_{\eta*} : G_{\rho,k} \longrightarrow G_{\rho,k+1}\}, \\ \sigma_\rho \text{ on } G_{\rho,\eta} &= \lim_{j}\{\lambda_{\eta*} : G_{j,\eta} \longrightarrow G_{j+1,\eta}\} \end{aligned}$$

by setting

$$\begin{aligned} \sigma_\rho :&[g,k] \in G_{\rho,k} \longrightarrow [g,k-1] \in G_{\rho,k-1}, \\ \sigma_\eta :&[g,j] \in G_{j,\eta} \longrightarrow [g,j-1] \in G_{j-1,\eta}. \end{aligned}$$

Lemma 7.12.

(i) *There exists an isomorphism* $\Phi_{\rho,\infty} : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow G_{\rho,\eta}$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ {\scriptstyle \Phi_{\rho,\infty}}\big\downarrow & & {\scriptstyle \Phi_{\rho,\infty}}\big\downarrow \\ G_{\rho,\eta} & \xrightarrow{\sigma_\eta} & G_{\rho,\eta} \end{array}$$

and hence

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{id-\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ {\scriptstyle \Phi_{\rho,\infty}}\big\downarrow & & {\scriptstyle \Phi_{\rho,\infty}}\big\downarrow \\ G_{\rho,\eta} & \xrightarrow{id-\sigma_\eta} & G_{\rho,\eta}. \end{array}$$

(ii) *There exists an isomorphism* $\Phi_{\infty,\eta} : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow G_{\rho,\eta}$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ {\scriptstyle \Phi_{\infty,\eta}}\big\downarrow & & {\scriptstyle \Phi_{\infty,\eta}}\big\downarrow \\ G_{\rho,\eta} & \xrightarrow{\sigma_\rho} & G_{\rho,\eta} \end{array}$$

and hence

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{id-\gamma_\rho^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ {\scriptstyle \Phi_{\infty,\eta}}\big\downarrow & & {\scriptstyle \Phi_{\infty,\eta}}\big\downarrow \\ G_{\rho,\eta} & \xrightarrow{id-\sigma_\rho} & G_{\rho,\eta}. \end{array}$$

As $J_{\mathcal{A}} : \mathcal{A} = \mathcal{F}_{0,0} \subset \mathcal{F}_{\rho,\eta}$ is a subalgebra, there exists a homomorphism

$$J_{\mathcal{A}*} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{F}_{\rho,\eta}).$$

Lemma 7.13. *The homomorphism $J_{\mathcal{A}*} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{F}_{\rho,\eta})$ is injective such that*

$$J_{A*} \circ \lambda_{\rho*} = \gamma_\rho^{-1} \circ J_{A*} \quad \textit{and} \quad J_{A*} \circ \lambda_{\eta*} = \gamma_\eta^{-1} \circ J_{A*}.$$

Proof. We will first show that the endomorphisms $\lambda_{\rho*}, \lambda_{\eta*}$ on $K_0(\mathcal{A})$ are both injective. Put a projection $Q_\alpha = S_\alpha S_\alpha^*$ and a subalgebra $\mathcal{A}_\alpha = \rho_\alpha(\mathcal{A})$ of $\mathcal{A}$ for $\alpha \in \Sigma^\rho$. Then the endomorphism ρ_α on $\mathcal{A}$ can extend to an isomorphism from $\mathcal{A}Q_\alpha$ onto $\mathcal{A}_\alpha$ by setting $\rho_\alpha(x) = S_\alpha^* x S_\alpha, x \in \mathcal{A}Q_\alpha$ whose inverse is $\phi_\alpha : \mathcal{A}_\alpha \longrightarrow \mathcal{A}Q_\alpha$ defined by $\phi_\alpha(y) = S_\alpha y S_\alpha^*, y \in \mathcal{A}_\alpha$. Hence the induced homomorphism $\rho_{\alpha*} : K_0(\mathcal{A}Q_\alpha) \longrightarrow K_0(\mathcal{A}_\alpha)$ is an isomorphism. Since $\mathcal{A} = \bigoplus_{\alpha\in\Sigma^\rho} Q_\alpha\mathcal{A}$, the homomorphism

$$\sum_{\alpha\in\Sigma^\rho} \phi_{\alpha*} \circ \rho_{\alpha*} : K_0(\mathcal{A}) \longrightarrow \bigoplus_{\alpha\in\Sigma^\rho} K_0(Q_\alpha\mathcal{A})$$

is an isomorphism, one may identify $K_0(\mathcal{A}) = \bigoplus_{\alpha\in\Sigma^\rho} K_0(Q_\alpha\mathcal{A})$. Let $g \in K_0(\mathcal{A})$ satisfy $\lambda_{\rho*}(g) = 0$. Put $g_\alpha = \phi_{\alpha*} \circ \rho_{\alpha*}(g) \in K_0(Q_\alpha\mathcal{A})$ for $\alpha \in \Sigma^\rho$ so that $g = \sum_{\alpha\in\Sigma^\rho} g_\alpha$. As $\rho_{\beta*} \circ \phi_{\alpha*} = 0$ for $\beta \neq \alpha$, one sees $\rho_{\beta*}(g_\alpha) = 0$ for $\beta \neq \alpha$. Hence we have

$$0 = \lambda_{\rho*}(g) = \sum_{\beta\in\Sigma^\rho} \sum_{\alpha\in\Sigma^\rho} \rho_{\beta*}(g_\alpha) = \sum_{\alpha\in\Sigma^\rho} \rho_{\alpha*}(g_\alpha) \in \bigoplus_{\alpha\in\Sigma^\rho} K_0(\mathcal{A}_\alpha).$$

It follows that $\rho_{\alpha*}(g_\alpha) = 0$ in $K_0(\mathcal{A}_\alpha)$. Since $\rho_{\alpha*} : K_0(Q_\alpha\mathcal{A}) \longrightarrow K_0(\mathcal{A}_\alpha)$ is isomorophic, one sees that $g_\alpha = 0$ in $K_0(\mathcal{A}Q_\alpha)$ for all $\alpha \in \Sigma^\rho$. This implies that $g = \sum_{\alpha\in\Sigma^\rho} g_\alpha = 0$ in $K_0(\mathcal{A})$. Therefore we know that the endomorphism $\lambda_{\rho*}$ on $K_0(\mathcal{A})$ is injective. and similarly so is $\lambda_{\eta*}$.

By the previous lemma, there exists an isomorphism $\Phi_{j,k} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{A})$ such that the following diagram

$$\begin{array}{ccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) \\
{\scriptstyle\Phi_{j,k}}\downarrow & & {\scriptstyle\Phi_{j+1,k}}\downarrow \\
K_0(\mathcal{A}) & \xrightarrow{\lambda_{\rho*}} & K_0(\mathcal{A})
\end{array}$$

is commutative so that the embedding $\iota_{+1,*} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j+1,k})$ is injective, and similarly $\iota_{*,+1} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j,k+1})$ is injective. Hence for $n, m \in \mathbb{N}$, the homomorphism

$$\iota_{n,m} : K_0(\mathcal{A}) = K_0(\mathcal{F}_{0,0}) \longrightarrow K_0(\mathcal{F}_{n,m})$$

defined by the compositions of $\iota_{+1,*}$ and $\iota_{*,+1}$ is injective. By [40, Theorem 6.3.2 (iii)], one knows $\mathrm{Ker}(J_{\mathcal{A}*}) = \cup_{n,m\in\mathbb{N}}\mathrm{Ker}(\iota_{n,m})$, so that $\mathrm{Ker}(J_{\mathcal{A}*}) = 0$. ☐

We henceforth identify the group $K_0(\mathcal{A})$ with its imaga $J_{A*}(K_0(\mathcal{A}))$ in $K_0(\mathcal{F}_{\rho,\eta})$. As in the above proof, not only $K_0(\mathcal{A})(= K_0(\mathcal{F}_{0,0}))$ but also the groups $K_0(\mathcal{F}_{j,k})$ for j, k are identified with subgroups of $K_0(\mathcal{F}_{\rho,\eta})$ via injective homomorphisms from $K_0(\mathcal{F}_{j,k})$ to $K_0(\mathcal{F}_{\rho,\eta})$ induced by the embedding of $\mathcal{F}_{j,k}$ into $\mathcal{F}_{\rho,\eta}$.

We note that

$$(\mathrm{id} - \gamma_\eta)(K_0(\mathcal{F}_{\rho,\eta})) = (\mathrm{id} - \gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta})),$$
$$(\mathrm{id} - \gamma_\rho)(K_0(\mathcal{F}_{\rho,\eta})) = (\mathrm{id} - \gamma_\rho^{-1})(K_0(\mathcal{F}_{\rho,\eta}))$$

and

$$(\mathrm{id} - \gamma_\rho) \cap (\mathrm{id} - \gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta}) = (\mathrm{id} - \gamma_\rho^{-1}) \cap (\mathrm{id} - \gamma_\eta^{-1}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}).$$

Denote by $(\mathrm{id}-\gamma_\rho)(K_0(\mathcal{F}_{\rho,\eta}))+(\mathrm{id}-\gamma_\eta)(K_0(\mathcal{F}_{\rho,\eta}))$ the subgroup of $K_0(\mathcal{F}_{\rho,\eta})$ generated by $(\mathrm{id}-\gamma_\rho)(K_0(\mathcal{F}_{\rho,\eta}))$ and $(\mathrm{id}-\gamma_\eta)(K_0(\mathcal{F}_{\rho,\eta}))$.

Lemma 7.14. *An element in $K_0(\mathcal{F}_{\rho,\eta})$ is equivalent to an element of $K_0(\mathcal{A})$ modulo the subgroup $(\mathrm{id}-\gamma_\rho)(K_0(\mathcal{F}_{\rho,\eta}))+(\mathrm{id}-\gamma_\eta)(K_0(\mathcal{F}_{\rho,\eta}))$.*

Proof. For $g\in K_0(\mathcal{F}_{\rho,\eta})$, we may assume that $g\in K_0(\mathcal{F}_{j,k})$ for some $j,k\in\mathbb{Z}_+$. As γ_ρ^{-1} commutes with γ_η^{-1}, one sees that $(\gamma_\rho^{-1})^j\circ(\gamma_\eta^{-1})^k(g)\in K_0(\mathcal{A})$. Put $g_1=\gamma_\rho^{-1}(g)$ so that

$$g-(\gamma_\rho^{-1})^j\circ(\gamma_\eta^{-1})^k(g)=g-\gamma_\rho^{-1}(g)+g_1-(\gamma_\rho^{-1})^{j-1}\circ(\gamma_\eta^{-1})^k(g_1).$$

We inductively see that $g-(\gamma_\rho^{-1})^j\circ(\gamma_\eta^{-1})^k(g)$ belongs to the subgroup $(\mathrm{id}-\gamma_\rho)(K_0(\mathcal{F}_{\rho,\eta}))+(\mathrm{id}-\gamma_\eta)(K_0(\mathcal{F}_{\rho,\eta}))$. Hence g is equivalent to $(\gamma_\rho^{-1})^j\circ(\gamma_\eta^{-1})^k(g)$ modulo $(\mathrm{id}-\gamma_\rho)(K_0(\mathcal{F}_{\rho,\eta}))+(\mathrm{id}-\gamma_\eta)(K_0(\mathcal{F}_{\rho,\eta}))$. □

Denote by $(\mathrm{id}-\lambda_{\rho*})(K_0(\mathcal{A}))+(\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A}))$ the subgroup of $K_0(\mathcal{A})$ generated by $(\mathrm{id}-\lambda_{\rho*})(K_0(\mathcal{A}))$ and $(\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A}))$.

Lemma 7.15. *For $g\in K_0(\mathcal{A})$, the condition $g\in(\mathrm{id}-\gamma_\rho^{-1})(K_0(\mathcal{F}_{\rho,\eta}))+(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta}))$ implies $g\in(\mathrm{id}-\lambda_{\rho*})(K_0(\mathcal{A}))+(\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A}))$.*

Proof. By the assumption that $g\in(\mathrm{id}-\gamma_\rho^{-1})(K_0(\mathcal{F}_{\rho,\eta}))+(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta}))$ there exist $g_1\in(\mathrm{id}-\gamma_\rho^{-1})(K_0(\mathcal{F}_{\rho,\eta}))$ and $g_2\in(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta}))$ such that $g=g_1+g_2$, where $g_1=(\mathrm{id}-\gamma_\rho^{-1})(h_1)$ and $g_2=(\mathrm{id}-\gamma_\eta^{-1})(h_2)$ for some $h_1,h_2\in K_0(\mathcal{F}_{\rho,\eta})$. We may assume that $h_1,h_2\in K_0(\mathcal{F}_{j,k})$ for large enough $j,k\in\mathbb{Z}_+$. Put $e_i=(\gamma_\rho^{-1})^j\circ(\gamma_\eta^{-1})^k(h_i)$ which belongs to $K_0(\mathcal{F}_{0,0})(=K_0(\mathcal{A}))$ for $i=0,1$. It follows that

$$\lambda_\rho^j\circ\lambda_\eta^k(g)=(\mathrm{id}-\lambda_{\eta*})(e_1)+(\mathrm{id}-\lambda_{\rho*})(e_2).$$

Now $g\in K_0(\mathcal{A})$ and $\lambda_\rho^j\circ\lambda_\eta^k(g)\in(\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A}))+(\mathrm{id}-\lambda_{\rho*})(K_0(\mathcal{A}))\subset K_0(\mathcal{A})$. As in the proof of the preceding lemma, by putting $g^{(n)}=\lambda_{\rho*}^n(g), g^{(n,m)}=\lambda_{\eta*}^m(g^{(n)})\in K_0(\mathcal{A})$ we have

$$\begin{aligned}
&g-\lambda_{\rho*}^j\circ\lambda_{\eta*}^k(g)\\
=&g-\lambda_{\rho*}(g)+g^{(1)}-\lambda_{\rho*}(g^{(1)})+g^{(2)}-\lambda_{\rho*}(g^{(2)})+\cdots+g^{(j-1)}-\lambda_{\rho*}(g^{(j-1)})\\
&+g^{(j)}-\lambda_{\eta*}^k(g^{(j)})\\
=&g-\lambda_{\rho*}(g)+g^{(1)}-\lambda_{\rho*}(g^{(1)})+g^{(2)}-\lambda_{\rho*}(g^{(2)})+\cdots+g^{(j-1)}-\lambda_{\rho*}(g^{(j-1)})\\
&+g^{(j)}-\lambda_{\eta*}(g^{(j)})+g^{(j,1)}-\lambda_{\eta*}(g^{(j,1)})+g^{(j,2)}-\lambda_{\eta*}(g^{(j,2)})+\cdots\\
&+g^{(j,k-1)}-\lambda_{\eta*}(g^{(j,k-1)})\\
=&(\mathrm{id}-\lambda_{\rho*})(g+g^{(1)}+\cdots+g^{(j-1)})+(\mathrm{id}-\lambda_{\eta*})(g^{(j)}+g^{(j,1)}+\cdots+g^{(j,k-1)})
\end{aligned}$$

Since $\lambda_{\rho*}^j\circ\lambda_{\eta*}^k(g)\in(\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A}))+(\mathrm{id}-\lambda_{\rho*})(K_0(\mathcal{A}))$ and

$$(\mathrm{id}-\lambda_{\rho*})(g+g^{(1)}+\cdots+g^{(j-1)})\in(\mathrm{id}-\lambda_{\rho*})(K_0(\mathcal{A})),$$
$$(\mathrm{id}-\lambda_{\eta*})(g^{(j)}+g^{(j,1)}+\cdots+g^{(j,k-1)})\in(\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A})),$$

we have

$$g\in(\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A}))+(\mathrm{id}-\lambda_{\rho*})(K_0(\mathcal{A})).$$

□

Hence we obtain the following lemma for the cokernel.

Lemma 7.16. *The quotient group*

$$K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta}))+(\mathrm{id}-\gamma_\rho^{-1})(K_0(\mathcal{F}_{\rho,\eta})))$$

is isomorphic to the quotient group

$$K_0(\mathcal{A})/((\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A}))+(\mathrm{id}-\lambda_{\rho*})(K_0(\mathcal{A})))$$

Proof. Surjectivity of the quotient map

$$q_{\mathcal{A}*}: K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta}))+(\mathrm{id}-\gamma_\rho^{-1})(K_0(\mathcal{F}_{\rho,\eta})))$$

comes from the preceding lemma. As

$$\mathrm{Ker}(q_{\mathcal{A}*}) = (\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A}))+(\mathrm{id}-\lambda_{\rho*})(K_0(\mathcal{A}))$$

by the preceing lemma, we have a desired assertion. □

For the kernel, we have

Lemma 7.17. *The subgroup*

$$\mathrm{Ker}(\mathrm{id}-\gamma_\eta^{-1})\cap\mathrm{Ker}(\mathrm{id}-\gamma_\rho^{-1}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})$$

is isomorphic to the subgroup

$$\mathrm{Ker}(\mathrm{id}-\lambda_{\eta*})\cap\mathrm{Ker}(\mathrm{id}-\lambda_{\rho*}) \text{ in } K_0(\mathcal{A})$$

through $J_{\mathcal{A}*}$.

Proof. For $g \in \mathrm{Ker}(\mathrm{id}-\gamma_\eta^{-1})\cap\mathrm{Ker}(\mathrm{id}-\gamma_\rho^{-1})$ in $K_0(\mathcal{F}_{\rho,\eta})$, one may assume that $g \in K_0(\mathcal{F}_{j,k})$ for some $j,k \in \mathbb{Z}_+$ so that $(\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) \in K_0(\mathcal{A})$. Through the identification between $J_{\mathcal{A}*}(K_0(\mathcal{A}))$ and $K_0(\mathcal{A})$ via $J_{\mathcal{A}*}$, one may assume that $\lambda_{\eta*} = \gamma_\eta^{-1}$ and $\lambda_{\rho*} = \gamma_\rho^{-1}$ on $K_0(\mathcal{A})$. As $g \in \mathrm{Ker}(\mathrm{id}-\gamma_\eta^{-1})\cap\mathrm{Ker}(\mathrm{id}-\gamma_\rho^{-1})$, one has $g = (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) \in K_0(\mathcal{A})$. This implies that $g \in \mathrm{Ker}(\mathrm{id}-\lambda_{\eta*})\cap\mathrm{Ker}(\mathrm{id}-\lambda_{\rho*})$ in $K_0(\mathcal{A})$. The converse inclusion relation $\mathrm{Ker}(\mathrm{id}-\lambda_{\eta*})\cap\mathrm{Ker}(\mathrm{id}-\lambda_{\rho*}) \subset \mathrm{Ker}(\mathrm{id}-\gamma_\eta^{-1})\cap\mathrm{Ker}(\mathrm{id}-\gamma_\rho^{-1})$ is clear through the above identification. □

Therefore we have

Proposition 7.18. *There exists a short exact sequence:*

$$\begin{aligned}0 &\longrightarrow K_0(\mathcal{A})/((\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A}))+(\mathrm{id}-\lambda_{\rho*})(K_0(\mathcal{A})))\\ &\longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta})\\ &\longrightarrow \mathrm{Ker}(\mathrm{id}-\lambda_{\eta*})\cap\mathrm{Ker}(\mathrm{id}-\lambda_{\rho*}) \text{ in } K_0(\mathcal{A}) \longrightarrow 0.\end{aligned}$$

Let $\mathcal{F}_\rho$ be the fixed point algebra $(\mathcal{O}_\rho)^{\hat\rho}$ of the C^*-symbolic dynamical system $(\mathcal{A},\rho,\Sigma^\rho)$. The algebra $\mathcal{F}_\rho$ is isomorphic to the subalgebra $\mathcal{F}_{\rho,0}$ of $\mathcal{F}_{\rho,\eta}$ in a natural way. As in the proof of Lemma 7.14, the group $K_0(\mathcal{F}_{\rho,0})$ is regarded as a subgroup of $K_0(\mathcal{F}_{\rho,\eta})$ and the restriction of γ_η^{-1} to $K_0(\mathcal{F}_{\rho,0})$ satisfies $\gamma_\eta^{-1}(K_0(\mathcal{F}_{\rho,0})) \subset K_0(\mathcal{F}_{\rho,0})$ so that γ_η^{-1} yields an endomorphism on $\mathcal{F}_\rho$, which we denote by γ_η^{-1}.

For the group $K_1(\mathcal{O}^\kappa_{\rho,\eta})$, we provide several lemmas. Their proofs are similarly to the above discussions.

Lemma 7.19.

(i) *An element in* $K_0(\mathcal{F}_{\rho,\eta})$ *is equivalent to an element of* $K_0(\mathcal{F}_{\rho,0})(= K_0(\mathcal{F}_\rho))$ *modulo the subgroup* $(\mathrm{id}-\gamma_\eta)(K_0(\mathcal{F}_{\rho,\eta}))$.

(ii) *If* $g \in K_0(\mathcal{F}_\rho)(= K_0(\mathcal{F}_{\rho,0}))$ *belongs to* $(\mathrm{id}-\gamma_\eta)(K_0(\mathcal{F}_{\rho,\eta}))$, *then* g *belongs to* $(\mathrm{id}-\gamma_\eta)(K_0(\mathcal{F}_\rho))$.

Lemma 7.20. *The quotient group* $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta}))$ *is isomorphic to the quotient group* $K_0(\mathcal{F}_\rho)/(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_\rho))$*, that is also isomorphic to the quotient group* $K_0(\mathcal{A})/(1-\lambda_\eta)K_0(\mathcal{A})$ *such that the kernel of* $\mathrm{id}-\gamma_\rho$ *in* $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta}))$ *is isomorphic to the kernel of* $\mathrm{id}-\lambda_\rho$ *in* $K_0(\mathcal{A})/(1-\lambda_\eta)K_0(\mathcal{A})$*. That is*

$$\begin{aligned}&\mathrm{Ker}(\mathrm{id}-\gamma_\rho) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta}))\\ \cong&\mathrm{Ker}(\mathrm{id}-\lambda_{\rho*}) \text{ in } K_0(\mathcal{A})/(\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A})).\end{aligned}$$

Proof. The first assertion that the three quotient groups

$$K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta})),\quad K_0(\mathcal{F}_\rho)/(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_\rho)),\quad K_0(\mathcal{A})/(1-\lambda_\eta)K_0(\mathcal{A})$$

are naturally isomorphic is similarly proved to the previous discussions. For the second assertion, the kernel $\mathrm{Ker}(\mathrm{id}-\gamma_\rho)$ in $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_{\rho,\eta}))$ is isomorphic to the kernel $\mathrm{Ker}(\mathrm{id}-\gamma_\rho)$ in $K_0(\mathcal{F}_\rho)/(\mathrm{id}-\gamma_\eta^{-1})(K_0(\mathcal{F}_\rho))$ which is isomorphic to the kernel $\mathrm{Ker}(\mathrm{id}-\lambda_{\rho*})$ in $K_0(\mathcal{A})/(1-\lambda_{\eta*})(K_0(\mathcal{A}))$. □

Lemma 7.21. *The kernel of* $\mathrm{id}-\gamma_\rho$ *in* $K_0(\mathcal{F}_{\rho,\eta})$ *is isomorphic to the kernel of* $\mathrm{id}-\gamma_\rho$ *in* $K_0(\mathcal{F}_\rho)$ *that is also isomorphic to the kernel of* $\mathrm{id}-\lambda_{\eta*}$ *in* $K_0(\mathcal{A})$ *such that the quotient group*

$$\mathrm{Ker}(\mathrm{id}-\gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\rho)(\mathrm{Ker}(\mathrm{id}-\gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))$$

is isomorphic to the quotient group

$$\mathrm{Ker}(\mathrm{id}-\lambda_{\eta*}) \text{ in } K_0(\mathcal{A})/(\mathrm{id}-\lambda_{\rho*})(\mathrm{Ker}(\mathrm{id}-\lambda_{\eta*}) \text{ in } K_0(\mathcal{A}).$$

That is

$$\begin{aligned}&\mathrm{Ker}(\mathrm{id}-\gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\rho)(\mathrm{Ker}(\mathrm{id}-\gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))\\ \cong&\mathrm{Ker}(\mathrm{id}-\lambda_{\eta*}) \text{ in } K_0(\mathcal{A})/(\mathrm{id}-\lambda_{\rho*})(\mathrm{Ker}(\mathrm{id}-\lambda_{\eta*}) \text{ in } K_0(\mathcal{A}).\end{aligned}$$

Proof. The proofs are similar to the previous discussions. □

Therefore we have

Proposition 7.22. *There exists a short exact sequence:*

$$\begin{aligned}0 &\longrightarrow \mathrm{Ker}(\mathrm{id}-\lambda_{\eta*}) \text{ in } K_0(\mathcal{A})/(\mathrm{id}-\lambda_{\rho*})(\mathrm{Ker}(\mathrm{id}-\lambda_{\eta*}) \text{ in } K_0(\mathcal{A}))\\ &\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta})\\ &\longrightarrow \mathrm{Ker}(\mathrm{id}-\lambda_{\rho*}) \text{ in } K_0(\mathcal{A})/(\mathrm{id}-\lambda_{\eta*})(K_0(\mathcal{A})) \longrightarrow 0.\end{aligned}$$

Consequently we have

Theorem 7.23. *Suppose that a C^*-textile dynamical system* $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$ *forms square. Then there exist short exact sequences for their K-theory groups as in the following way:*

$$\begin{aligned}0 &\longrightarrow K_0(\mathcal{A})/(\mathrm{id}-\lambda_{\eta*})K_0(\mathcal{A})+(\mathrm{id}-\lambda_{\rho*})K_0(\mathcal{A})\\ &\longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta})\\ &\longrightarrow \mathrm{Ker}(\mathrm{id}-\lambda_{\eta*})\cap\mathrm{Ker}(\mathrm{id}-\lambda_{\rho*}) \text{ in } K_0(\mathcal{A}) \longrightarrow 0\end{aligned}$$

and

$$\begin{aligned}0 &\longrightarrow \mathrm{Ker}(\mathrm{id} - \lambda_{\eta *}) \textit{ in } K_0(\mathcal{A})/(\mathrm{id} - \lambda_{\rho *})(\mathrm{Ker}(\mathrm{id} - \lambda_{\eta *}) \textit{ in } K_0(\mathcal{A}))\\ &\longrightarrow K_1(\mathcal{O}^{\kappa}_{\rho,\eta})\\ &\longrightarrow \mathrm{Ker}(\mathrm{id} - \lambda_{\rho *}) \textit{ in } K_0(\mathcal{A})/(\mathrm{id} - \lambda_{\eta *})K_0(\mathcal{A}) \longrightarrow 0\end{aligned}$$

where the endomorphisms $\lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ *are defined by*

$$\lambda_{\rho *}([p]) = \sum_{\alpha\in\Sigma^\rho} [\rho_{\alpha *}(p)] \in K_0(\mathcal{A}) \textit{ for } [p] \in K_0(\mathcal{A}),$$

$$\lambda_{\eta *}([p]) = \sum_{a\in\Sigma^\eta} [\eta_{a *}(p)] \in K_0(\mathcal{A}) \textit{ for } [p] \in K_0(\mathcal{A}).$$

8. Examples

1. LR-textile λ-graph systems.

A symbolic matrix $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ is a matrix whose components consist of formal sums of elements of Σ, such as

$$\mathcal{M} = \begin{bmatrix} a & a+c \\ c & 0 \end{bmatrix} \qquad \text{where } \Sigma = \{a,b,c\}.$$

$\mathcal{M}$ is said to be essential if there is no zero column or zero row. $\mathcal{M}$ is said to be left-resolving if for each column a symbol does not appear in two different rows. For example, $\begin{bmatrix} a & a+b \\ c & 0 \end{bmatrix}$ is left-resolving, but $\begin{bmatrix} a & a+b \\ c & b \end{bmatrix}$ is not left-resolving because of b at the second column. We henceforth asssume that symbolic matrices are always essential and left-resolving.

Let $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ and $\mathcal{M}' = [\mathcal{M}'(i,j)]_{i,j=1}^N$ be symbolic matrices over Σ and Σ' respectively. Suppose that there is a bijection $\kappa : \Sigma \longrightarrow \Sigma'$. Following Nasu's terminology [29] we say that $\mathcal{M}$ and $\mathcal{M}'$ are equivalent under specification κ, or simply a specified equivalence if $\mathcal{M}'$ can be obtained from $\mathcal{M}$ by replacing every symbol $\alpha \in \Sigma$ by $\kappa(\alpha)$. That is if $\mathcal{M}(i,j) = \alpha_1 + \cdots + \alpha_n$, then $\mathcal{M}'(i,j) = \kappa(\alpha_1) + \cdots + \kappa(\alpha_n)$. We write this situation as $\mathcal{M} \overset{\kappa}{\cong} \mathcal{M}'$ (see [29]).

For a symbolic matrix $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ over $\Sigma^{\mathcal{M}}$, we set for $\alpha \in \Sigma^{\mathcal{M}}, i,j = 1,\dots,N$

$$A^{\mathcal{M}}(i,\alpha,j) = \begin{cases} 1 & \text{if } \alpha \text{ appears in } \mathcal{M}(i,j), \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

Put an $N \times N$ nonnegative matrix $A^{\mathcal{M}} = [A_{\mathcal{M}}(i,j)]_{i,j=1}^N$ by setting $A^{\mathcal{M}}(i,j) = \sum_{\alpha\in\Sigma^{\mathcal{M}}} A^{\mathcal{M}}(i,\alpha,j)$. Let $\mathcal{A}$ be an N-dimensional commutative C^*-algebra $\mathbb{C}^N$ with minimal projections $E_1,\dots,E_n$ such that

$$\mathcal{A} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n.$$

We set for $\alpha \in \Sigma^{\mathcal{M}}$:

$$\rho^{\mathcal{M}}_\alpha(E_i) = \sum_{j=1}^N A^{\mathcal{M}}(i,\alpha,j)E_j, \qquad i = 1,\dots,n.$$

Then we have a C^*-symbolic dynamical system $(\mathcal{A}, \rho^{\mathcal{M}}, \Sigma^{\mathcal{M}})$.

Let $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ and $\mathcal{N} = [\mathcal{N}(i,j)]_{i,j=1}^N$ be symbolic matrices over $\Sigma^{\mathcal{M}}$ and $\Sigma^{\mathcal{N}}$ respectively. Suppose that there is a bijection $\kappa : \Sigma^{\mathcal{M}} \longrightarrow \Sigma^{\mathcal{N}}$ such that κ yields a specified equivalence

$$\mathcal{M}\mathcal{N} \overset{\kappa}{\cong} \mathcal{N}\mathcal{M}. \tag{8.2}$$

Then we have two C^*-symbolic dynamical systems $(\mathcal{A}, \rho^{\mathcal{M}}, \Sigma^{\mathcal{M}})$ and $(\mathcal{A}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{N}})$. Put

$$\Sigma^{\mathcal{M}\mathcal{N}} = \{(\alpha, b) \in \Sigma^{\mathcal{M}} \times \Sigma^{\mathcal{N}} \mid \rho_b^{\mathcal{N}} \circ \rho_\alpha^{\mathcal{M}} \neq 0\},$$
$$\Sigma^{\mathcal{N}\mathcal{M}} = \{(a, \beta) \in \Sigma^{\mathcal{N}} \times \Sigma^{\mathcal{M}} \mid \rho_\beta^{\mathcal{M}} \circ \rho_a^{\mathcal{N}} \neq 0\}.$$

Proposition 8.1. *Keep the above situations. κ induces a specification $\kappa : \Sigma^{\mathcal{M}\mathcal{N}} \longrightarrow \Sigma^{\mathcal{N}\mathcal{M}}$ such that*

$$\rho_b^{\mathcal{N}} \circ \rho_\alpha^{\mathcal{M}} = \rho_\beta^{\mathcal{M}} \circ \rho_a^{\mathcal{N}} \quad \textit{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{8.3}$$

Hence $(\mathcal{A}, \rho^{\mathcal{M}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{M}}, \Sigma^{\mathcal{N}}, \kappa)$ yields a C^*-textile dynamical system.

Proof. Since $\mathcal{M}\mathcal{N} \overset{\kappa}{\cong} \mathcal{N}\mathcal{M}$, one sees that for $i, j = 1, 2, \dots, N$, $\kappa(\mathcal{M}\mathcal{N}(i,j)) = \mathcal{N}\mathcal{M}(i,j)$. For $(\alpha, b) \in \Sigma^{\mathcal{M}\mathcal{N}}$, there exists $i, k = 1, 2, \dots, N$ such that $\rho_b^{\mathcal{N}} \circ \rho_\alpha^{\mathcal{M}}(E_i) \geq E_k$. As $\kappa(\alpha, b)$ appears in $\mathcal{N}\mathcal{M}(i,k)$, by putting $(a, \beta) = \kappa(\alpha, b)$, we have $\rho_\beta^{\mathcal{M}} \circ \rho_a^{\mathcal{N}}(E_i) \geq E_k$. Hence $\kappa(\alpha, b) \in \Sigma^{\mathcal{N}\mathcal{M}}$. One indeed sees that $\rho_b^{\mathcal{N}} \circ \rho_\alpha^{\mathcal{M}} = \rho_\beta^{\mathcal{M}} \circ \rho_a^{\mathcal{N}}$ by the relation $\mathcal{M}\mathcal{N} \overset{\kappa}{\cong} \mathcal{N}\mathcal{M}$. $\square$

We remark that symbolic matrices are presentations of labeled directed graphs. Hence we may consider our discussions above in terms of labeled directed graphs.

Two symbolic matrices satisfying the relations (8.2) gives rise to LR textile systems that have been introduced by Nasu (see [29]). Textile systems introduced by Nasu play a important tool to analyze automorphisms and endomorphisms of topological Markov shifts,

The author has generalized the LR-textile systems to LR-textile λ-graph systems which consists of two commuting symbolic matrix systems ([25]). Let $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ be an LR-textile λ-graph system defined by a specified equivalence:

$$\mathcal{M}_{l,l+1}\mathcal{N}_{l+1,l+2} \overset{\kappa}{\cong} \mathcal{N}_{l,l+1}\mathcal{M}_{l+1,l+2}, \qquad l \in \mathbb{Z}_+ \tag{8.4}$$

through specification κ. There exist two symbolic matrix systems $(\mathcal{M}, I^{\mathcal{M}})$ and $(\mathcal{N}, I^{\mathcal{N}})$. Denote by $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ the associated λ-graph systems respectively. Since $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ form square in the sense of [25, p.170], they have common sequences $V_l^{\mathcal{M}} = V_l^{\mathcal{N}}, l \in \mathbb{Z}_+$ of vertices and inclusion matrices $I_{l,l+1}^{\mathcal{M}} = I_{l,l+1}^{\mathcal{N}}, l \in \mathbb{Z}_+$. We denote $V_l^{\mathcal{M}} = V_l^{\mathcal{N}}$ and $I_{l,l+1}^{\mathcal{M}} = I_{l,l+1}^{\mathcal{N}}$ by V_l and $I_{l,l+1}$ respectively.

Let $(\mathcal{A}_{\mathcal{M}}, \rho^{\mathcal{M}}, \Sigma^{\mathcal{M}})$ and $(\mathcal{A}_{\mathcal{N}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{N}})$ be the associated C^*-symbolic dynamical systems with the λ-graph systems $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ respectively. Hence one sees that $\mathcal{A}_{\mathcal{M}} = \mathcal{A}_{\mathcal{N}}$ which is denoted by $\mathcal{A}$. It is easy to seee that the relation (9.1) implies

$$\rho_\alpha^{\mathcal{M}} \circ \rho_b^{\mathcal{N}} = \rho_a^{\mathcal{N}} \circ \rho_\beta^{\mathcal{M}} \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{8.5}$$

Proposition 8.2. *An LR-textile λ-graph system $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ yields a C^*-textile dynamical system $(\mathcal{A}, \rho^{\mathcal{M}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{M}}, \Sigma^{\mathcal{N}}, \kappa)$ which forms square.*

Conversely, a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which forms square yields an LR-textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}^\rho}_{\mathcal{M}^\eta}}$ such that the associated C^*-textile dynamical system $(\mathcal{A}, \rho^{\mathcal{M}^\rho}, \rho^{\mathcal{M}^\eta}, \Sigma^{\mathcal{M}^\rho}, \Sigma^{\mathcal{M}^\rho}, \kappa)$ is a subsystem of $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ in the sense which satisfis the following relations:

$$\mathcal{A}_{\mathfrak{L}} \subset \mathcal{A}, \qquad \rho|_{\mathcal{A}^{\mathfrak{L}}} = \rho^{\mathcal{M}^\rho}, \qquad \eta|_{\mathcal{A}^{\mathfrak{L}}} = \rho^{\mathcal{M}^\eta}.$$

Proof. Let $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ be an LR-textile λ-graph system. As in the above discussions, we have a C^*-textile dynamical system $(\mathcal{A}, \rho^{\mathcal{M}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{M}}, \Sigma^{\mathcal{N}}, \kappa)$.

Conversely, let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-textile dynamical system which forms square. Put for $l \in \mathbb{N}$

$$\mathcal{A}_l^\rho = C^*(\rho_\mu(1) : \mu \in B_l(\Lambda_\rho)), \qquad \mathcal{A}_l^\eta = C^*(\eta_\xi(1) : \xi \in B_l(\Lambda_\eta)).$$

Since $\mathcal{A}_l^\rho = \mathcal{A}_l^\eta$ and they are commutative and of finite dimensional, the algebra

$$\mathcal{A}_{\mathfrak{L}} = \overline{\cup_{l \in \mathbb{Z}_+} \mathcal{A}_l^\rho} = \overline{\cup_{l \in \mathbb{Z}_+} \mathcal{A}_l^\eta}$$

is a commutative AF-subalgebra of $\mathcal{A}$. It is easy to see that both $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ are C^*-symbolic dynamical systems such that

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta) \tag{8.6}$$

By construction, there exist λ-graph systems $\mathfrak{L}^\rho$ and $\mathfrak{L}^\eta$ whose C^*-symbolic dynamical systems are $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ respectively. Let $(\mathcal{M}^\rho, I^\rho)$ and $(\mathcal{M}^\eta, I^\eta)$ be the associated symbolic dynamical systems. It is easy to see that the relation (**??**) implies

$$\mathcal{M}^\rho_{l,l+1}\mathcal{M}^\eta_{l+1,l+2} \overset{\kappa}{\cong} \mathcal{M}^\eta_{l,l+1}\mathcal{M}^\rho_{l+1,l+2}, \qquad l \in \mathbb{Z}_+. \tag{8.7}$$

Hence we have an LR-textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}^\rho}_{\mathcal{M}^\eta}}$. It is direct to see that the associated C^*-textile dynamical system is

$$(\mathcal{A}_{\mathfrak{L}}, \rho|_{\mathcal{A}_{\mathfrak{L}}}, \eta|_{\mathcal{A}_{\mathfrak{L}}}, \Sigma^\rho, \Sigma^\eta, \kappa).$$

□

Let A be an $N \times N$ matrix with entries in nonnegative integers. We may consider a directed graph $G_A = (V_A, E_A)$ with vertex set V_A and edge set E_A. The vertex set V_A consists of N vertices which we denote by $\{v_1, \dots, v_N\}$. We equip $A(i,j)$ edges from the vertex v_i to the vertex v_j. Denote by E_A such edges. Let $\Sigma^A = E_A$ and the labeling map $\lambda_A : E_A \longrightarrow \Sigma^A$ be defined as the identity map. Then we have a labeled directed graph denoted by G_A as well as a symbolic matrix $\mathcal{M}_A = [\mathcal{M}_A(i,j)]_{i,j=1}^N$ by setting

$$\mathcal{M}_A(i,j) = \begin{cases} e_1 + \cdots + e_n & \text{if } e_1, \cdots, e_n \text{ are edges from } v_i \text{ to } v_j, \\ 0 & \text{if there is no edge from } v_i \text{ to } v_j. \end{cases}$$

Let B be an $N \times N$ matrix with entries in nonnegative integers such that

$$AB = BA.$$

Hence the numbers of pairs of directed edges

$$\{(e,f) \in E_A \times E_B \mid s(e) = v_i, t(e) = s(f), t(f) = v_k\}$$
$$\{(f,e) \in E_B \times E_A \mid s(f) = v_i, t(f) = s(e), t(e) = v_k\}.$$

coincide with each other for each v_i and v_k, so that one may take a bijection $\kappa : \Sigma^{AB} \longrightarrow \Sigma^{BA}$ which gives rise to a specified equivalence $\mathcal{M}_A\mathcal{M}_B \overset{\kappa}{\cong} \mathcal{M}_A\mathcal{M}_B$. Therefore we have a C^*-textile dynamical system

$$(\mathcal{A}, \rho^{\mathcal{M}_A}, \rho^{\mathcal{M}_B}, \Sigma^A, \Sigma^B, \kappa)$$

which we denote by

$$(\mathcal{A}, \rho^A, \rho^B, \Sigma^A, \Sigma^B, \kappa).$$

The associated C^*-algebra is denoted by $\mathcal{O}^\kappa_{A,B}$. We remark that the algebra $\mathcal{O}^\kappa_{A,B}$ is dependent on the choice of specification $\kappa : \Sigma^{AB} \longrightarrow \Sigma^{BA}$. The algebras are 2-graph algebras of Kumjian and Pask [16]. They are C^*-algebras associated to textile systems studied by V. Deaconu [8].

Proposition 8.3. *Keep the above situations. There exist short exact sequences:*

$$\begin{aligned} 0 &\longrightarrow \mathbb{Z}^N/((1-A)\mathbb{Z}^N + (1-B)\mathbb{Z}^N) \\ &\longrightarrow K_0(\mathcal{O}^\kappa_{A,B}) \\ &\longrightarrow \operatorname{Ker}(1-A) \cap \operatorname{Ker}(1-B) \ \textit{in}\ \mathbb{Z}^N \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\longrightarrow \operatorname{Ker}(1-B) \ \textit{in}\ \mathbb{Z}^N/(1-A)(\operatorname{Ker}(1-B) \ \textit{in}\ \mathbb{Z}^N) \\ &\longrightarrow K_1(\mathcal{O}^\kappa_{A,B}) \\ &\longrightarrow \operatorname{Ker}(1-A) \ \textit{in} \quad \mathbb{Z}^N/(1-B)\mathbb{Z}^N \longrightarrow 0. \end{aligned}$$

We consider 1×1 matrices $[N]$ and $[M]$ with its entries N and M respectively for $1 < N, M \in \mathbb{N}$. Let G_N be a directed labeled graph with one vertex and N-self directed loops. Similarly we consider a directed labeled graph G_M with M-self loops at the vertex. Denote by $\Sigma^N = \{f_1, \dots, f_N\}$ the set of directed N-self loops of G_N and $\Sigma^M = \{e_1, \dots, e_M\}$ the set of directed M-self loops of G_M. The correspondence $(e, f) \in \Sigma^M \times \Sigma^N \longrightarrow (f, e) \in \Sigma^N \times \Sigma^M$ yields a specification κ, which we will fix. Put

$$\rho^M_{e_i}(1) = 1, \qquad \rho^N_{f_j}(1) = 1.$$

Then we have a C^*-textile dynamical system

$$(\mathbb{C}, \rho^M, \rho^N, \Sigma^M, \Sigma^N, \kappa).$$

The associated C^*-algebra is denoted by $\mathcal{O}_{M,N}$.

Lemma 8.4. $\mathcal{O}_{N,M} = \mathcal{O}_N \otimes \mathcal{O}_M$.

Proof. Let $s_i, i = 1, \dots, N$ and $t_j, i = 1, \dots, M$ and be the generating isometries of $\mathcal{O}_N$ and of $\mathcal{O}_M$ satisfying

$$\sum_{i=1}^N s_i s_i^* = 1, \qquad \sum_{j=1}^M t_j t_j^* = 1$$

Let $S_i, i = 1, \dots, N$ and $T_j, i = 1, \dots, M$ and be the generating isometries of $\mathcal{O}_{N,M}$ satisfying

$$\sum_{i=1}^N S_i S_i^* = 1, \qquad \sum_{j=1}^M T_j T_j^* = 1$$

such that

$$S_iT_j = T_jS_i, \qquad i = 1, \dots, N, \quad j = 1, \dots, M.$$

Since $(s_i \otimes 1)(1 \otimes t_j) = (1 \otimes t_j)(s_i \otimes 1), i = 1, \dots, N, \quad j = 1, \dots, M$, By the universality of $\mathcal{O}_{N,M}$ subject to the relations, one has a surjective homomorphism $\Phi : \mathcal{O}_{N,M} \longrightarrow \mathcal{O}_N \otimes \mathcal{O}_M$ such that $\Phi(S_i) = s_i \otimes 1, \ \Phi(T_j) = 1 \otimes t_j$. And also by the universality of the tensor product $\mathcal{O}_N \otimes \mathcal{O}_M$, there exists a homomorphism $\Psi : \mathcal{O}_N \otimes \mathcal{O}_M \longrightarrow \mathcal{O}_{N,M}$ such that $\Psi(s_i \otimes 1) = S_i, \ \Psi(1 \otimes t_j) = T_j$. Since $\Phi \circ \Psi = \mathrm{id}, \Psi \circ \Phi = \mathrm{id}$, one concludes that Φ and Ψ are inverses to each other so that $\mathcal{O}_{N,M} \cong \mathcal{O}_N \otimes \mathcal{O}_M$. As both $\mathcal{O}_N$ and $\mathcal{O}_M$ are simple, purely infinite, we have $\mathcal{O}_{N,M}$ is simple, purely infinite. □

Lemma 8.5. *Put $n = N - 1, m = M - 1$. Then we have the subgroup $\{[k] \in \mathbb{Z}/m\mathbb{Z} \mid nk \in m\mathbb{Z}\}$ of $\mathbb{Z}/m\mathbb{Z}$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$.*

Proof. As $d = g.c.d(n, m)$, there exist $n_0, m_0 \in \mathbb{N}$ such that $m = m_0d, n = n_0d$ and $(n_0, m_0) = 1$. For $k \in \mathbb{Z}$ with $nk \in m\mathbb{Z}$, the condition $(n_0, m_0) = 1$ implies $k = m_0k'$ for some $k' \in \mathbb{Z}$. Hence $k \in m_0\mathbb{Z}$ so that we see that the subgroup $\{[k] \in \mathbb{Z}/m\mathbb{Z} \mid nk \in m\mathbb{Z}\}$ of $\mathbb{Z}/m\mathbb{Z}$ is isomorphic to $m_0\mathbb{Z}/m_0d\mathbb{Z}$, which is isomorphic to $\mathbb{Z}/d\mathbb{Z}$. □

Lemma 8.6. *For $1 < N, M \in \mathbb{N}$ with $d = g.c.d(N - 1, M - 1)$,*

(i) $\mathbb{Z}/((N - 1)\mathbb{Z} + (N - 1)\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$.
(ii) $\mathrm{Ker}(N - 1) = \mathrm{Ker}(M - 1) = 0$ *in* $\mathbb{Z}$.

Proof. It is easy to show that the subgroup $(N - 1)\mathbb{Z} + (N - 1)\mathbb{Z}$ of $\mathbb{Z}$ coincides with $d\mathbb{Z}$. (ii) is trivial. □

Therefore we have

Proposition 8.7. *For $1 < N, M \in \mathbb{N}$, the C^*-algebra $\mathcal{O}_{N,M}$ is simple, purely infinite, such that*

$$K_0(\mathcal{O}_{N,M}) \cong K_1(\mathcal{O}_{N,M}) \cong \mathbb{Z}/d\mathbb{Z}$$

where $d = g.c.d(N - 1, M - 1)$ the greatest common diviser of $N - 1, M - 1$.

It is easy to see that the K-groups $K_i(\mathcal{O}_N \otimes \mathcal{O}_M)$ are $\mathbb{Z}/d\mathbb{Z}$ for $i = 0, 1$ by using the Künneth formula proved in [42].

We will generalize the above examles from the view point of tensor products.

2. Tensor products.

Let $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ be C^*-symbolic dynamical systems. We will construct a C^*-textile dynamical system as follows: Put

$$\bar{\mathcal{A}} = \mathcal{A}^\rho \otimes \mathcal{A}^\eta, \qquad \bar{\rho}_\alpha = \rho_\alpha \otimes \mathrm{id}, \qquad \bar{\eta}_a = \mathrm{id} \otimes \eta_a, \qquad \Sigma^{\bar{\rho}} = \Sigma^\rho, \qquad \Sigma^{\bar{\eta}} = \Sigma^\eta$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, where $\otimes$ means the minimal C^*-tensor product $\otimes_{\min}$. For $(\alpha, a) \in \Sigma^\rho \times \Sigma^\eta$, we see $\eta_b \circ \rho_\alpha(1) \neq 0$ if and only if $\eta_b(1) \neq 0, \rho_\alpha(1) \neq 0$, so that

$$\Sigma_{\bar{\rho}\bar{\eta}} = \Sigma^\rho \times \Sigma^\eta \quad \text{ and similarly } \quad \Sigma_{\bar{\eta}\bar{\rho}} = \Sigma^\eta \times \Sigma^\rho.$$

Lemma 8.8. *Define $\bar{\kappa} : \Sigma_{\bar{\rho}\bar{\eta}} \longrightarrow \Sigma_{\bar{\eta}\bar{\rho}}$ by setting $\bar{\kappa}(\alpha, b) = (b, \alpha)$. We then have $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is a C^*-textile dynamical system.*

Proof. By [1], we have $Z_{\bar{\mathcal{A}}} = Z_{\mathcal{A}^\rho} \otimes Z_{\mathcal{A}^\eta}$ so that

$$\bar{\rho}_\alpha(Z_{\bar{\mathcal{A}}}) \subset Z_{\bar{\mathcal{A}}}, \quad \alpha \in \Sigma^{\bar{\rho}} \quad \text{and} \quad \bar{\rho}_a(Z_{\bar{\mathcal{A}}}) \subset Z_{\bar{\mathcal{A}}}, \quad a \in \Sigma^{\bar{\eta}}.$$

We also have $\sum_{\alpha\in\Sigma^{\bar{\rho}}} \bar{\rho}_\alpha(1) = \sum_{\alpha\in\Sigma^{\rho}} \rho_\alpha(1) \otimes 1 \geq 1$, and similarly $\sum_{a\in\Sigma^{\bar{\eta}}} \bar{\eta}_(1) \geq 1$ so that both families $\{\bar{\rho}_\alpha\}_{\alpha\in\Sigma^{\bar{\rho}}}\}$ and $\{\bar{\eta}_a\}_{a\in\Sigma^{\bar{\eta}}}\}$ of endomorphisms are essential. Since $\{\rho_\alpha\}_{\alpha\in\Sigma^\rho}$ is faithful on $\mathcal{A}^\rho$, the homomorphism

$$x \in \mathcal{A}^\rho \longrightarrow \oplus_{\alpha\in\Sigma^\rho}\rho_\alpha(x) \in \oplus_{\alpha\in\Sigma^\rho}\mathcal{A}^\rho$$

is injective so that the homomorphism

$$x \otimes y \in \mathcal{A}^\rho \otimes \mathcal{A}^\eta \longrightarrow \oplus_{\alpha\in\Sigma^\rho}\rho_\alpha(x) \otimes y \in \oplus_{\alpha\in\Sigma^\rho}\mathcal{A}^\rho \otimes \mathcal{A}^\eta$$

is injective. This implies that $\{\bar{\rho}_\alpha\}_{\alpha\in\Sigma^{\bar{\rho}}}$ is faithful and similary so is $\{\bar{\eta}_a\}_{a\in\Sigma^{\bar{\eta}}}$. Hence $(\bar{\mathcal{A}}, \bar{\rho}, \Sigma^{\bar{\rho}})$ and $(\bar{\mathcal{A}}, \bar{\eta}, \Sigma^{\bar{\eta}})$ are C^*-symbolic dynamical systems. It is direct to see that $\bar{\eta}_b \circ \bar{\rho}_\alpha = \bar{\rho}_\alpha \circ \bar{\eta}_b$ for $(\alpha, b) \in \Sigma_{\bar{\rho}\bar{\eta}}$. Therefore $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is a C^*-textile dynamical system. □

We call $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ the tensor product between $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$. Denote by $S_\alpha, \alpha \in \Sigma^{\bar{\rho}}, T_a, a \in \Sigma^{\bar{\eta}}$ the generating partial isometries of the C^*-algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ for the C^*-textile dynamical system $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$. By the universality for the algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ subject to the relations $(\bar{\rho}, \bar{\eta}, \bar{\kappa})$, one sees that the algebra $\mathcal{D}_{\bar{\rho},\bar{\eta}}$ is isomorphic to the tensor product $\mathcal{D}_\rho \otimes \mathcal{D}_\eta$ through the correspondence

$$S_\mu T_\xi (x \otimes y) T_\xi^* S_\mu^* \longleftrightarrow S_\mu x S_\mu^* \otimes T_\xi y T_\xi^*$$

for $\mu \in B_*(\Lambda_\rho), \xi \in B_*(\Lambda_\eta),\ x \in A^\rho, y \in \mathcal{A}^\eta$.

Lemma 8.9. *Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free (resp. AF-free). Then the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is free (resp. AF-free).*

Proof. Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free. There exist increasing sequences $\mathcal{A}^\rho_l, l \in \mathbb{Z}_+$ and $\mathcal{A}^\eta_l, l \in \mathbb{Z}_+$ of C^*-subalgebras of $\mathcal{A}^\rho$ and $\mathcal{A}^\eta$ satisfying the conditions of the freeness respectively. Put $\bar{\mathcal{A}}_l = \mathcal{A}^\rho_l \otimes \mathcal{A}^\eta_l, l \in \mathbb{Z}_+$ It is clear that

(1) $\bar{\rho}_\alpha(\bar{\mathcal{A}}_l) \subset \bar{\mathcal{A}}_{l+1}, \alpha \in \Sigma^{\bar{\rho}}$ and $\bar{\eta}_a(\bar{\mathcal{A}}_l) \subset \bar{\mathcal{A}}_{l+1}, a \in \Sigma^{\bar{\eta}}$ for $l \in \mathbb{Z}_+$.

(2) $\cup_{l\in\mathbb{Z}_+}\bar{\mathcal{A}}_l$ is dense in $\bar{\mathcal{A}}$.

We will show that the condition (3) in the definition of freeness holds. Take and fix arbitrary $j, k, l \in \mathbb{N}$ with $j + k \leq l$. For $j \leq l$, by the freeness of $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ there exists a projection $q_\rho \in \mathcal{D}_\rho \cap \mathcal{A}^{\rho\prime}_l$ such that

(i) $q_\rho x \neq 0$ for $0 \neq x \in \mathcal{A}^\rho_l$,

(ii) $\phi^n_\rho(q_\rho) q_\rho = 0$ for all $n = 1, 2, \ldots, j$.

Similarly for $k \leq l$, by the freeness of $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ there exists a projection $q_\eta \in \mathcal{D}_\eta \cap \mathcal{A}^{\eta\prime}_l$ such that

(i) $q_\eta y \neq 0$ for $0 \neq y \in \mathcal{A}^\eta_l$,

(ii) $\phi^m_\eta(q_\eta) q_\eta = 0$ for all $m = 1, 2, \ldots, k$.

Put $q = q_\rho \otimes q_\eta \in \mathcal{D}_\rho \otimes \mathcal{D}_\eta (= \mathcal{D}_{\bar{\rho},\bar{\eta}})$ so that $q \in \mathcal{D}_{\bar{\rho},\bar{\eta}} \cap \bar{\mathcal{A}}'_l$. As the maps $\Phi^\rho_l : x \in \mathcal{A}^\rho_l \longrightarrow q_\rho x \in q_\rho \mathcal{A}^\rho_l$ and $\Phi^\eta_l : y \in \mathcal{A}^\eta_l \longrightarrow q_\eta x \in q_\eta \mathcal{A}^\eta_l$ are isomorphisms so that the tensor product

$$\Phi^\rho_l \otimes \Phi^\eta_l : x \otimes y \in \mathcal{A}^\rho_l \otimes \mathcal{A}^\eta_l \longrightarrow (q_\rho \otimes q_\eta)(x \otimes y) \in (q_\rho \otimes q_\eta) \in \mathcal{A}^\rho_l \otimes \mathcal{A}^\eta_l$$

is isomorphic. Hence $qa \neq 0$ for $0 \neq a \in \bar{\mathcal{A}}_l$. It is straightforward to see that $\phi^n_\rho(q)\phi^m_\eta(q) = \phi^n_\rho((\phi^m_\eta(q)))q = \phi^n_\rho(q)q = \phi^m_\eta(q)q = 0$ for all $n = 1, 2, \ldots, j$, $m =$

$1, 2, \dots, k$. Therefore the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is free. It is obvious to see that if both $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are AF-free, then $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is AF-free. □

Proposition 8.10. *Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free. Then the C^*-algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ for the tensor product C^*-textile dynamical system $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is isomorphic to the tensor product $\mathcal{O}_\rho \otimes \mathcal{O}_\eta$ of the C^*-algebras between $\mathcal{O}_\rho$ and $\mathcal{O}_\eta$. If in particular, $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both irreducible, the C^*-algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ is simple.*

Proof. Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free. By the preceding lemma, the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is free and hence satisfies condition (I). Let $s_\alpha, \alpha \in \Sigma^\rho$ and $t_a, a \in \Sigma^\eta$ be the generating partial isometries of the C^*-algebras $\mathcal{O}_\rho$ and $\mathcal{O}_\eta$ respectively. Let $S_\alpha, \alpha \in \Sigma^{\bar{\rho}}$ and $T_a, a \in \Sigma^{\bar{\eta}}$ be the generating partial isometries of the C^*-algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$. By the uniqueness of the algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ with respect to the relations $(\bar{\rho}, \bar{\eta}, \bar{\kappa})$, the correspondence

$$S_\alpha \longrightarrow s_\alpha \otimes 1 \in \mathcal{O}_\rho \otimes \mathcal{O}_\eta, \qquad T_a \longrightarrow 1 \otimes t_a \in \mathcal{O}_\rho \otimes \mathcal{O}_\eta$$

naturally gives rise to an isomorphism from $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ onto the tensor product $\mathcal{O}_\rho \otimes \mathcal{O}_\eta$.

If in particular, $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both irreducible, the C^*-algebras $\mathcal{O}_\rho$ and $\mathcal{O}_\eta$ are both simple so that $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ is simple. □

We remark that the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ does not necessarily form square. The K-theory groups $K_*(\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}})$ are computed from the Künneth formulae for $K_*(\mathcal{O}_\rho \otimes \mathcal{O}_\eta)$ [42].

In [28], higher dimensional analogue $(\mathcal{A}, \rho^1, \dots, \rho^N, \Sigma^1, \dots, \Sigma^N, \kappa)$ will be studied.

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Department of Mathematics, Joetsu University of Education, Joetsu 943-8512, Japan
E-mail address: kengo@juen.ac.jp

C^*-ALGEBRAS ASSOCIATED WITH TEXTILE DYNAMICAL SYSTEMS

KENGO MATSUMOTO

ABSTRACT. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of a C^*-algebra $\mathcal{A}$ with some conditions. It yields a C^*-algebra $\mathcal{O}_\rho$ from an associated Hilbert C^*-bimodule. In this paper, we will extend the notion of C^*-symbolic dynamical system to C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which consists of two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with certain commutation relations κ between their endomorphisms $\{\rho_\alpha\}_{\alpha\in\Sigma^\rho}$ and $\{\eta_a\}_{a\in\Sigma^\eta}$. C^*-textile dynamical systems yield two-dimensional subshifts and C^*-algebras $\mathcal{O}^\kappa_{\rho,\eta}$. We will study the structure of the algebras $\mathcal{O}^\kappa_{\rho,\eta}$ and present its K-theory formulae.

1. Introduction

In [20], the author has introduced a notion of λ-graph system as presentations of subshifts. The λ-graph systems are labeled Bratteli diagram with shift transformation. They yield C^*-algebras so that its K-theory groups are related to topological conjugacy invariants of the underlying symbolic dynamical systems. The class of these C^*-algebras include the Cuntz–Krieger algebras. He has extended the notion of λ-graph system to C^*-*symbolic dynamical system*, which is a generalization of both a λ-graph system and an automorphism of a unital C^*-algebra. It is a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of a unital C^*-algebra $\mathcal{A}$ such that $\rho_\alpha(Z_\mathcal{A}) \subset Z_\mathcal{A}, \alpha \in \Sigma$ and $\sum_{\alpha\in\Sigma} \rho_\alpha(1) \geq 1$ where $Z_\mathcal{A}$ denotes the center of $\mathcal{A}$. A finite labeled graph $\mathcal{G}$ gives rise to a C^*-symbolic dynamical system $(\mathcal{A}_\mathcal{G}, \rho^\mathcal{G}, \Sigma)$ such that $\mathcal{A} = \mathbb{C}^N$ for some $N \in \mathbb{N}$. A λ-graph system $\mathfrak{L}$ is a generalization of a finite labeled graph and yields a C^*-symbolic dynamical system $(\mathcal{A}_\mathfrak{L}, \rho^\mathfrak{L}, \Sigma)$ such that $\mathcal{A}_\mathfrak{L}$ is $C(\Omega_\mathfrak{L})$ for some compact Hausdorff space $\Omega_\mathfrak{L}$ with $\dim\Omega_\mathfrak{L} = 0$. It also yields a C^*-algebra $\mathcal{O}_\mathfrak{L}$. A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ provides a subshift Λ_ρ over Σ and a Hilbert C^*-bimodule $\mathcal{H}^\rho_\mathcal{A}$ over $\mathcal{A}$. The C^*-algebra $\mathcal{O}_\rho$ for $(\mathcal{A}, \rho, \Sigma)$ may be realized as a Cuntz-Pimsner algebra from the Hilbert C^*-bimodule $\mathcal{H}^\rho_\mathcal{A}$ ([23], cf. [12], [34]). We call the algebra $\mathcal{O}_\rho$ *the C^*-symbolic crossed product of $\mathcal{A}$ by the subshift* Λ_ρ. If $\mathcal{A} = C(X)$ with $\dim X = 0$, there exists a λ-graph system $\mathfrak{L}$ such that the subshift Λ_ρ is the subshift $\Lambda_\mathfrak{L}$ presented by $\mathfrak{L}$ and the C^*-algebra $\mathcal{O}_\rho$ is the C^*-algebra $\mathcal{O}_\mathfrak{L}$ associated with $\mathfrak{L}$. If in particular, $\mathcal{A} = \mathbb{C}^N$, the subshift Λ_ρ is a sofic shift and $\mathcal{O}_\rho$ is a Cuntz–Krieger algebra. If $\Sigma = \{\alpha\}$ an automorphism α of a unital C^*-algebra $\mathcal{A}$, the C^*-algebra $\mathcal{O}_\rho$ is the ordinary crossed product $\mathcal{A} \times_\alpha \mathbb{Z}$.

G. Robertson–T. Steger [37] have initiated a certain study of higher dimensional analogue of Cuntz–Krieger algebras from the view point of tiling systems of 2-dimensional plane. After their work, A. Kumjian–D. Pask [15] have generalized their construction to introduce the notion of higher rank graphs and its C^*-algebras. The C^*-algebras constructed from higher rank graphs are called the higher rank

graph C^*-algebras. Since then, there have been many studies on these C^*-algebras by many authors (see for example [8], [9], [15], [35], [30], [37], etc.).

M. Nasu in [28] has introduced the notion of textile system which is useful in analyzing automorphisms and endomorphisms of topological Markov shifts. A textile system also gives rise to a two-dimensional tiling called Wang tiling. Among textile systems, LR textile systems have specific properties that consist of two commuting symbolic matrices. In [24], the author has extended the notion of textile systems to λ-graph systems and has defined a notion of textile systems on λ-graph systems, which are called textile λ-graph systems for short. C^*-algebras associated to textile systems have been initiated by V. Deaconu ([8]).

In this paper, we will extend the notion of C^*-symbolic dynamical system to C^*-textile dynamical system which is a higher dimensional analogue of C^*-symbolic dynamical system. The C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ consists of two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with the following commutation relations between ρ and η through κ. Set

$$\Sigma^{\rho\eta} = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_\alpha \neq 0\}, \quad \Sigma^{\eta\rho} = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_\beta \circ \eta_a \neq 0\}.$$

We require that there exists a bijection $\kappa : \Sigma^{\rho\eta} \longrightarrow \Sigma^{\eta\rho}$, which we fix and call a specification. Then the required commutation relations are

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{1.1}$$

A C^*-textile dynamical system provides a two-dimensional subshifts and a C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$. The C^*-algbebra $\mathcal{O}^\kappa_{\rho,\eta}$ is defined to be the universal C^*-algebra $C^*(x, S_\alpha, T_a; x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta)$ generated by $x \in \mathcal{A}$ and two families of partial isometries $S_\alpha, \alpha \in \Sigma^\rho$, $T_a, a \in \Sigma^\eta$ subject to the following relations called $(\rho, \eta; \kappa)$:

$$\sum_{\beta \in \Sigma^\rho} S_\beta S_\beta^* = 1, \qquad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x), \tag{1.2}$$

$$\sum_{b \in \Sigma^\eta} T_b T_b^* = 1, \qquad x T_a T_a^* = T_a T_a^* x, \qquad T_a^* x T_a = \eta_a(x), \tag{1.3}$$

$$S_\alpha T_b = T_a S_\beta \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta) \tag{1.4}$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$.

In Section 3, we will construct a tiling system in the plane from a C^*-textile dynamical system. The resulting tiling system is a two-dimensional subshift. In Section 4, we will study some basic properties of the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$. In Section 5, we will introduce a condition called (I) on $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which will be studied as a generalization of the condition (I) on C^*-symbolic dynamical system [22](cf. [7], [21]). We will show the following

Theorem 1.1. *Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-textile dynamical system satisfying condition (I). Then the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ is the unique C^*-algebra subject to the relations $(\rho, \eta; \kappa)$. If $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is irreducible, $\mathcal{O}^\kappa_{\rho,\eta}$ is simple.*

In Section 6, we will realize $\mathcal{O}^\kappa_{\rho,\eta}$ as a Cuntz-Pimsner algebra associated with a certain Hilbert C^*-bimodule. A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to form square if the C^*-subalgebra of $\mathcal{A}$ generated by the projections $\rho_\alpha(1), \alpha \in \Sigma^\rho$ and the C^*-subalgebra of $\mathcal{A}$ generated by the projections $\eta_a(1), a \in \Sigma^\eta$ coincide. In Section 7 and 8, we will restrict our interest to the C^*-textile dynamical systems forming square to prove the following K-theory formulae:

Theorem 1.2. *Suppose that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ froms square. There exist short exact sequences for $K_0(\mathcal{O}^\kappa_{\rho,\eta})$ and $K_1(\mathcal{O}^\kappa_{\rho,\eta})$ such that*

$$
\begin{aligned}
0 &\longrightarrow K_0(\mathcal{A})/((\mathrm{id}-\lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id}-\lambda_\rho)K_0(\mathcal{A}))\\
&\longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta})\\
&\longrightarrow \mathrm{Ker}(\mathrm{id}-\lambda_\eta)\cap\mathrm{Ker}(\mathrm{id}-\lambda_\rho) \text{ in } K_0(\mathcal{A}) \longrightarrow 0
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\longrightarrow (\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \text{ in } K_0(\mathcal{A}))/(\mathrm{id}-\lambda_\rho)(\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \text{ in } K_0(\mathcal{A}))\\
&\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta})\\
&\longrightarrow \mathrm{Ker}(\mathrm{id}-\bar{\lambda}_\rho) \text{ in } (K_0(\mathcal{A})/(\mathrm{id}-\lambda_\eta)K_0(\mathcal{A})) \longrightarrow 0
\end{aligned}
$$

where the endomorphisms $\lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ are defined by

$$
\begin{aligned}
\lambda_\rho([p]) &= \sum_{\alpha\in\Sigma^\rho}[\rho_\alpha(p)] \in K_0(\mathcal{A}) \text{ for } [p]\in K_0(\mathcal{A}),\\
\lambda_\eta([p]) &= \sum_{a\in\Sigma^\eta}[\eta_a(p)] \in K_0(\mathcal{A}) \text{ for } [p]\in K_0(\mathcal{A})
\end{aligned}
$$

and $\bar{\lambda}_\rho$ denotes an endomorphism on $K_0(\mathcal{A})/(1-\lambda_\eta)K_0(\mathcal{A})$ induced by λ_ρ.

Let A, B be mutually commuting $N \times N$ matrices with entries in nonnegative integers. Let $G_A = (V_A, E_A), G_B = (V_B, E_B)$ be directed graphs with common vertex set $V_A = V_B$, whose transition matrices are A, B respectively. Let $\mathcal{M}_A, \mathcal{M}_B$ denote symbolic marices for G_A, G_B whose components consist of formal sums of the directed edges of G_A, G_B respectively. Let Σ^{AB}, Σ^{BA} be the sets of the pairs of the concatenated directed edges in $E_A \times E_B, E_B \times E_A$ respectively. By the condition $AB = BA$, one may take a bijection $\kappa : \Sigma^{AB} \longrightarrow \Sigma^{BA}$ which gives rise to a specified equivalence $\mathcal{M}_A\mathcal{M}_B \overset{\kappa}{\cong} \mathcal{M}_A\mathcal{M}_B$. We then have a C^*-textile dynamical system written as

$$(\mathcal{A}, \rho^A, \rho^B, \Sigma^A, \Sigma^B, \kappa).$$

The associated C^*-algebra is denoted by $\mathcal{O}^\kappa_{A,B}$. The C^*-algebra $\mathcal{O}^\kappa_{A,B}$ is realized as a 2-graph C^*-algebra constructed from Kumjian–Pask ([15]). It is also seen in Deaconu's paper [8]. We will see the following proposition in Section 9.

Proposition 1.3. *Keep the above situations. There exist short exact sequences for $K_0(\mathcal{O}^\kappa_{A,B})$ and $K_1(\mathcal{O}^\kappa_{A,B})$ such that*

$$
\begin{aligned}
0 &\longrightarrow \mathbb{Z}^N/((1-A)\mathbb{Z}^N + (1-B)\mathbb{Z}^N)\\
&\longrightarrow K_0(\mathcal{O}^\kappa_{A,B})\\
&\longrightarrow \mathrm{Ker}(1-A)\cap\mathrm{Ker}(1-B) \text{ in } \mathbb{Z}^N \longrightarrow 0
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\longrightarrow (\mathrm{Ker}(1-B) \text{ in } \mathbb{Z}^N)/(1-A)(\mathrm{Ker}(1-B) \text{ in } \mathbb{Z}^N)\\
&\longrightarrow K_1(\mathcal{O}^\kappa_{A,B})\\
&\longrightarrow \mathrm{Ker}(1-\bar{A}) \text{ in } (\mathbb{Z}^N/(1-B)\mathbb{Z}^N) \longrightarrow 0,
\end{aligned}
$$

where $\bar{A}$ is an endomorphism on the abelian group $\mathbb{Z}^N/(1-B)\mathbb{Z}^N$ induced by the matrix A.

Throughout the paper, we will denote by $\mathbb{Z}_+$ the set of nonnegative integers and by $\mathbb{N}$ the set of positive integers.

2. λ-graph systems, C^*-symbolic dynamical systems and their C^*-algebras

In this section, we will briefly review λ-graph systems and C^*-symbolic dynamical systems. Throughout the section, Σ denotes a finite set with its discrete topology, that is called an alphabet. Each element of Σ is called a symbol. Let $\Sigma^{\mathbb{Z}}$ be the infinite product space $\prod_{i\in\mathbb{Z}}\Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}$ given by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$ is called the full shift over Σ. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a two-sided subshift, written as Λ for brevity.

There is a class of subshifts called sofic shifts, that are presented by finite labeled graphs. λ-graph systems are generalization of finite labeled graphs. Any subshift is presented by a λ-graph system. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ with vertex set $V = \cup_{l\in\mathbb{Z}_+}V_l$ and edge set $E = \cup_{l\in\mathbb{Z}_+}E_{l,l+1}$ that is labeled with symbols in Σ by a map $\lambda : E \to \Sigma$, and that is supplied with surjective maps $\iota(=\iota_{l,l+1}) : V_{l+1} \to V_l$ for $l \in \mathbb{Z}_+$. Here the vertex sets $V_l, l \in \mathbb{Z}_+$ and the edge sets $E_{l,l+1}, l \in \mathbb{Z}_+$ are finite disjoint sets for each $l \in \mathbb{Z}_+$. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1} respectively. Every vertex in V has a successor and every vertex in V_l for $l \in \mathbb{N}$ has a predecessor. It is then required that for vertices $u \in V_{l-1}$ and $v \in V_{l+1}$, there exists a bijective correspondence between the set of edges $e \in E_{l,l+1}$ such that $t(e) = v, \iota(s(e)) = u$ and the set of edges $f \in E_{l-1,l}$ such that $s(f) = u, t(f) = \iota(v)$, preserving thier labels ([20]). We assume that $\mathfrak{L}$ is left-resolving, which means that $t(e) \neq t(f)$ whenever $\lambda(e) = \lambda(f)$ for $e, f \in E_{l,l+1}$. Let us denote by $\{v_1^l, \dots, v_{m(l)}^l\}$ the vertex set V_l at level l. For $i = 1, 2, \dots, m(l),\ j = 1, 2, \dots, m(l+1),\ \alpha \in \Sigma$ we put

$$A_{l,l+1}(i,\alpha,j) = \begin{cases} 1 & \text{if } \ s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$
$$I_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } \ \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise.} \end{cases}$$

The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with $\mathfrak{L}$ is the universal C^*-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E_i^l, i = 1, 2, \dots, m(l),\ l \in \mathbb{Z}_+$ subject to the following operator relations called $(\mathfrak{L})$:

$$\sum_{\beta\in\Sigma} S_\beta S_\beta^* = 1, \tag{2.1}$$

$$\sum_{i=1}^{m(l)} E_i^l = 1, \qquad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) E_j^{l+1}, \tag{2.2}$$

$$S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*, \tag{2.3}$$

$$S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j) E_j^{l+1}, \tag{2.4}$$

for $i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+, \alpha \in \Sigma$. If $\mathfrak{L}$ satisfies λ-condition (I) and is λ-irreducible, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite ([22], [21]).

Let $\mathcal{A}_{\mathfrak{L},l}$ be the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the projections $E_i^l, i = 1, \ldots, m(l)$. We denote by $\mathcal{A}_{\mathfrak{L}}$ the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the all projections $E_i^l, i = 1, \ldots, m(l), l \in \mathbb{Z}_+$. As $\mathcal{A}_{\mathfrak{L},l} \subset \mathcal{A}_{\mathfrak{L},l+1}$ and $\cup_{l \in \mathbb{Z}_+} \mathcal{A}_{\mathfrak{L},l}$ is dense in $\mathcal{A}$, the algebra $\mathcal{A}_{\mathfrak{L}}$ is a commutative AF-algebra. For $\alpha \in \Sigma$, put

$$\rho_\alpha^{\mathfrak{L}}(X) = S_\alpha^* X S_\alpha \qquad \text{for} \qquad X \in \mathcal{A}_{\mathfrak{L}}.$$

Then $\{\rho_\alpha^{\mathfrak{L}}\}_{\alpha \in \Sigma}$ yields a family of $*$-endomorphisms of $\mathcal{A}_{\mathfrak{L}}$ such that $\rho_\alpha^{\mathfrak{L}}(1) \neq 0$, $\sum_{\alpha \in \Sigma} \rho_\alpha^{\mathfrak{L}}(1) \geq 1$ and for any nonzero $x \in \mathcal{A}_{\mathfrak{L}}$, $\rho_\alpha^{\mathfrak{L}}(x) \neq 0$ for some $\alpha \in \Sigma$.

The situations above are generalized to C^*-symbolic dynamical systems as follows. Let $\mathcal{A}$ be a unital C^*-algebra. In what follows, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit $1_{\mathcal{A}}$ of $\mathcal{A}$. The unit $1_{\mathcal{A}}$ is denoted by 1 unless we specify. Denote by $Z_{\mathcal{A}}$ the center of $\mathcal{A}$. Let $\rho_\alpha, \alpha \in \Sigma$ be a finite family of endomorphisms of $\mathcal{A}$ indexed by symbols of a finite set Σ. We assume that $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}, \alpha \in \Sigma$. The family $\rho_\alpha, \alpha \in \Sigma$ of endomorphisms of $\mathcal{A}$ is said to be *essential* if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and $\sum_\alpha \rho_\alpha(1) \geq 1$. It is said to be *faithful* if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$.

Definition (cf. [23]**).** A C^**-symbolic dynamical system* is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital C^*-algebra $\mathcal{A}$ and an essential and faithful finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of $\mathcal{A}$.

As in the above discussion, we have a C^*-symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ from a λ-graph system $\mathfrak{L}$. In [23], [25],[26], we have defined a C^*-symbolic dynamical system in a less restrictive way than the above definition. Instead of the above condition $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ with $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}, \alpha \in \Sigma$, we have used the condition in the papers that the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with $\mathcal{A}$. All of the examples appeared in the papers [23], [25], [26] satisfy the condition $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ with $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}, \alpha \in \Sigma$, and all discussions in the papers well work under the above new definition.

A C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift Λ_ρ over Σ such that a word $\alpha_1 \cdots \alpha_k$ of Σ is admissible for Λ_ρ if and only if $(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$ ([23, Proposition 2.1]). Denote by $B_k(\Lambda_\rho)$ the set of admissible words of Λ_ρ respectively with length k. Put $B_*(\Lambda_\rho) = \cup_{k=0}^\infty B_k(\Lambda_\rho)$, where $B_0(\Lambda_\rho)$ denotes the empty word. We say that *a subshift* Λ *acts on a* C^**-algebra* $\mathcal{A}$ if there exists a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift Λ_ρ is Λ.

The C^*-algebra $\mathcal{O}_\rho$ associated with a C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ has been originally constructed in [23] as a C^*-algebra by using the Pimsner's general construction of C^*-algebras from Hilbert C^*-bimodules [34] (cf. [12] etc.). It is realized as the universal C^*-algebra $C^*(x, S_\alpha; x \in \mathcal{A}, \alpha \in \Sigma)$ generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma$ subject to the following relations called (ρ):

$$\sum_{\beta \in \Sigma} S_\beta S_\beta^* = 1, \qquad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. Furthermore for $\alpha_1, \ldots, \alpha_k \in \Sigma$, a word $(\alpha_1, \ldots, \alpha_k)$ is admissible for the subshift Λ_ρ if and only if $S_{\alpha_1} \cdots S_{\alpha_k} \neq 0$ ([23, Proposition 3.1]). The C^*-algebra $\mathcal{O}_\rho$ is a generalization of the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$.

3. C^*-textile dynamical systems and two-dimensional subshifts

Let Σ be a finite set. The two-dimensional full shift over Σ is defined to be

$$\Sigma^{\mathbb{Z}^2} = \{(x_{i,j})_{(i,j)\in\mathbb{Z}^2} \mid x_{i,j} \in \Sigma\}.$$

An element $x \in \Sigma^{\mathbb{Z}^2}$ is regarded as a function $x : \mathbb{Z}^2 \longrightarrow \Sigma$ which is called a configuration on $\mathbb{Z}^2$. For $x \in \Sigma^{\mathbb{Z}^2}$ and $F \subset \mathbb{Z}^2$, let x_F denote the restriction of x to F. For a vector $m = (m_1, m_2) \in \mathbb{Z}^2$, let $\sigma^m : \Sigma^{\mathbb{Z}^2} \longrightarrow \Sigma^{\mathbb{Z}^2}$ be the translation along vector m defined by

$$\sigma^m((x_{i,j})_{(i,j)\in\mathbb{Z}^2}) = (x_{i+m_1,j+m_2})_{(i,j)\in\mathbb{Z}^2}.$$

A subset $X \subset \Sigma^{\mathbb{Z}^2}$ is said to be translation invariant if $\sigma^m(X) = X$ for all $m \in \mathbb{Z}^2$. It is obvious to see that a subset $X \subset \Sigma^{\mathbb{Z}^2}$ is translation invariant if ond only if X is invariant only both horizontaly and vertically, that is, $\sigma^{(1,0)}(X) = X$ and $\sigma^{(0,1)}(X) = X$. For $k \in \mathbb{Z}_+$, put

$$[-k,k]^2 = \{(i,j) \in \mathbb{Z}^2 \mid -k \le i,j \le k\} = [-k,k] \times [-k,k].$$

A metric d on $\Sigma^{\mathbb{Z}^2}$ is defined by for $x, y \in \Sigma^{\mathbb{Z}^2}$ with $x \ne y$

$$d(x,y) = \frac{1}{2^k} \quad \text{if} \quad x_{(0,0)} = y_{(0,0)},$$

where $k = \max\{k \in \mathbb{Z}_+ \mid x_{[-k,k]^2} = y_{[-k,k]^2}\}$. If $x_{(0,0)} \ne y_{(0,0)}$, put $k = -1$ on the above definition. If $x = y$, we set $d(x,y) = 0$. A two-dimensional subshift X is defined to be a closed, translation invariant subset of $\Sigma^{\mathbb{Z}^2}$ (cf. [17, p.467]). A finite subset $F \subset \mathbb{Z}^2$ is said to be a shape. A pattern f on a shape F is a function $f : F \longrightarrow \Sigma$. For a list $\mathfrak{F}$ of patterns, put

$$X_{\mathfrak{F}} = \{(x_{i,j})_{(i,j)\in\mathbb{Z}^2} \mid \sigma^m(x)|_F \notin \mathfrak{F} \text{ for all } m \in \mathbb{Z}^2 \text{ and } F \subset \mathbb{Z}^2\}.$$

It is well-known that a subset $X \subset \Sigma^{\mathbb{Z}^2}$ is a two-dimensional subshift if and only if there exists a list of patterns $\mathfrak{F}$ such that $X = X_{\mathfrak{F}}$.

We will define a certain property of two-dimensional subshift as follows:

Definition. A two-dimensional subshift X is said to have the *diagonal property* if for $(x_{i,j})_{(i,j)\in\mathbb{Z}^2}, (y_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X$, the conditions $x_{i,j} = y_{i,j}, x_{i+1,j-1} = y_{i+1,j-1}$ imply $x_{i,j-1} = y_{i,j-1}, x_{i+1,j} = y_{i+1,j}$. A two-dimensional subshift having the diagonal property is called *a textile dynamical system.*

Lemma 3.1. *If a two dimensional subshift X has the diagonal property, then for $x \in X$ and $(i,j) \in \mathbb{Z}^2$, the configuration x is determined by the diagonal line $(x_{i+n,j-n})_{n\in\mathbb{Z}}$ through (i,j).*

Proof. By the diagonal property, the sequence $(x_{i+n,j-n})_{n\in\mathbb{Z}}$ determines both the sequences $(x_{i+1+n,j-n})_{n\in\mathbb{Z}}$ and $(x_{i-1+n,j-n})_{n\in\mathbb{Z}}$. Repeating this way, the sequence $(x_{i+n,j-n})_{n\in\mathbb{Z}}$ determines the whole configuration x. $\square$

Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-textile dynamical system. It consists of two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with common unital C^*-algebra $\mathcal{A}$ and commutation relations between their endomorphisms $\rho_\alpha, \alpha \in \Sigma^\rho, \eta_a, a \in \Sigma^\eta$ through a bijection κ between the following sets $\Sigma^{\rho\eta}$ and $\Sigma^{\eta\rho}$, where

$$\Sigma^{\rho\eta} = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_\alpha \ne 0\}, \quad \Sigma^{\eta\rho} = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_\beta \circ \eta_a \ne 0\}.$$

The given bijection $\kappa : \Sigma^{\rho\eta} \longrightarrow \Sigma^{\eta\rho}$ is called a specification. The required commutation relations are

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{3.1}$$

A C^*-textile dynamical system will yield a two-dimensional subshift $X^\kappa_{\rho,\eta}$. We set

$$\Sigma_\kappa = \{\omega = (\alpha, b, a, \beta) \in \Sigma^\rho \times \Sigma^\eta \times \Sigma^\eta \times \Sigma^\rho \mid \kappa(\alpha, b) = (a, \beta)\}.$$

For $\omega = (\alpha, b, a, \beta)$, since $\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a$ as endomorphism on $\mathcal{A}$, one may identify the quadruplet (α, b, a, β) with the endomorphism $\eta_b \circ \rho_\alpha (= \rho_\beta \circ \eta_a)$ on $\mathcal{A}$ which we will denote by simply ω. Define maps $t(= top), b(= bottom) : \Sigma_\kappa \longrightarrow \Sigma^\rho$ and $l(= left), r(= right) : \Sigma_\kappa \longrightarrow \Sigma^\rho$ by setting

$$t(\omega) = \alpha, \quad b(\omega) = \beta, \quad l(\omega) = a, \quad r(\omega) = b.$$

$$\begin{tikzcd}
\cdot \arrow[r, "\alpha=t(\omega)"] \arrow[d, "a=l(\omega)"'] & \cdot \arrow[d, "b=r(\omega)"] \\
\cdot \arrow[r, "\beta=b(\omega)"'] & \cdot
\end{tikzcd}$$

A configuration $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_\kappa^{\mathbb{Z}^2}$ is said to be *paived* if the conditions

$$t(\omega_{i,j}) = b(\omega_{i,j+1}), \quad r(\omega_{i,j}) = l(\omega_{i+1,j}), \quad l(\omega_{i,j}) = r(\omega_{i-1,j}), \quad b(\omega_{i,j}) = t(\omega_{i,j-1})$$

hold for all $(i,j) \in \mathbb{Z}^2$. We set

$$\begin{aligned} X^\kappa_{\rho,\eta} = &\{(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_\kappa^{\mathbb{Z}^2} \mid (\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \text{ is paved and} \\ &\quad \omega_{i+n,j-n} \circ \omega_{i+n-1,j-n+1} \circ \cdots \circ \omega_{i+1,j-1} \circ \omega_{i,j} \neq 0 \text{ for all } (i,j) \in \mathbb{Z}^2, n \in \mathbb{N}\}, \end{aligned}$$

where $\omega_{i+n,j-n} \circ \omega_{i+n-1,j-n+1} \circ \cdots \circ \omega_{i+1,j-1} \circ \omega_{i,j}$ is the compositions as endomorphisms on $\mathcal{A}$.

Lemma 3.2. *Suppose that a configuration $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_\kappa^{\mathbb{Z}^2}$ is paved. Then $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^\kappa_{\rho,\eta}$ if and only if*

$$\rho_{b(\omega_{i+n,j-m})} \circ \cdots \circ \rho_{b(\omega_{i+1,j-m})} \circ \rho_{b(\omega_{i,j-m})} \circ \eta_{l(\omega_{i,j-m})} \circ \cdots \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})} \neq 0$$

for all $(i,j) \in \mathbb{Z}^2$, $n, m \in \mathbb{Z}_+$.

$$\begin{tikzcd}
\cdot \arrow[d, "l(\omega_{i,j})"'] & & & & \\
\cdot \arrow[d, "l(\omega_{i,j-1})"'] & & & & \\
\cdot & & & & \\
\vdots & & & & \\
\cdot \arrow[d, "l(\omega_{i,j-m})"'] & & & & \\
\cdot \arrow[r, "b(\omega_{i,j-m})"'] & \cdot \arrow[r, "b(\omega_{i+1,j-m})"'] & \cdots \arrow[r, "b(\omega_{i+n,j-m})"'] & \cdot &
\end{tikzcd}$$

Proof. Suppose that $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^{\kappa}_{\rho,\eta}$. For $(i,j) \in \mathbb{Z}^2$, $n,m \in \mathbb{Z}_+$, we may assume that $m \ge n$. Since

$$\begin{aligned}
0 \ne & \omega_{i+m,j-m} \circ \cdots \circ \omega_{i+n+1,j-m} \circ \omega_{i+n,j-m} \circ \cdots \circ \omega_{i,j-m} \circ \cdots \circ \omega_{i+1,j-1} \circ \omega_{i,j} \\
= & \omega_{i+m,j-m} \circ \cdots \circ \omega_{i+n+1,j-m} \circ \rho_{b(\omega_{i+n,j-m})} \circ \cdots \circ \rho_{b(\omega_{i+1,j-m})} \circ \rho_{b(\omega_{i,j-m})} \\
& \circ \eta_{l(\omega_{i,j-m})} \cdots \circ \eta_{l(\omega_{i,j-m})} \circ \cdots \circ \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})},
\end{aligned}$$

one has

$$\rho_{b(\omega_{i+n,j-m})} \circ \cdots \circ \rho_{b(\omega_{i+1,j-m})} \circ \rho_{b(\omega_{i,j-m})} \circ \eta_{l(\omega_{i,j-m})} \circ \cdots \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})} \ne 0.$$

The converse implication is clear by the equality:

$$\begin{aligned}
& \omega_{i+n,j-n} \circ \cdots \circ \omega_{i,j-n} \circ \cdots \circ \omega_{i,j-1} \circ \omega_{i,j} \\
= & \rho_{b(\omega_{i+n,j-n})} \circ \cdots \circ \rho_{b(\omega_{i,j-n})} \circ \eta_{l(\omega_{i,j-n})} \cdots \circ \eta_{l(\omega_{i,j-1})} \circ \eta_{l(\omega_{i,j})}.
\end{aligned}$$

□

Proposition 3.3. *$X^{\kappa}_{\rho,\eta}$ is a two-dimensional subshift having diagonal property, that is, $X^{\kappa}_{\rho,\eta}$ is a textile dynamical system.*

Proof. It is easy to see that the set

$$E = \{(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_{\kappa}^{\mathbb{Z}^2} \mid (\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \text{ is paved } \}$$

is closed, because its complement is open in $\Sigma_{\kappa}^{\mathbb{Z}^2}$. The following set

$$\begin{aligned}
U = \{(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \Sigma_{\kappa}^{\mathbb{Z}^2} \mid & \omega_{k+n,l-n} \circ \omega_{k+n-1,l-n+1} \circ \cdots \circ \omega_{k+1,l-1} \circ \omega_{k,l} = 0 \\
& \text{for some } (k,l) \in \mathbb{Z}^2, n \in \mathbb{N}\}
\end{aligned}$$

is open in $\Sigma_{\kappa}^{\mathbb{Z}^2}$. As the equality $X^{\kappa}_{\rho,\eta} = E \cap U^c$ holds, the set $X^{\kappa}_{\rho,\eta}$ is closed. It is also obvious that $X^{\kappa}_{\rho,\eta}$ is translation invariant so that $X^{\kappa}_{\rho,\eta}$ is a two-dimensional subshift. It is easy to see that $X^{\kappa}_{\rho,\eta}$ has diagonal property. □

We call $X^{\kappa}_{\rho,\eta}$ the textile dynamical system associated with $(\mathcal{A}, \rho, \eta, \Sigma^{\rho}, \Sigma^{\eta}, \kappa)$.

Let us now define a (one-dimensional) subshift $X_{\delta^{\kappa}}$ over Σ_{κ}, which consists of diagonal sequences of $X^{\kappa}_{\rho,\eta}$ as follows:

$$X_{\delta^{\kappa}} = \{(\omega_{n,-n})_{n\in\mathbb{Z}} \in \Sigma_{\kappa}^{\mathbb{Z}} \mid (\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^{\kappa}_{\rho,\eta}\}.$$

By Lemma 3.1, an element $(\omega_{n,-n})_{n\in\mathbb{Z}}$ of $X_{\delta^{\kappa}}$ may be extended to $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^{\kappa}_{\rho,\eta}$ in a unique way. Hence the one-dimensional subshift $X_{\delta^{\kappa}}$ determines the two-dimensional subshift $X^{\kappa}_{\rho,\eta}$. Therefore we have

Lemma 3.4. *The two-dimensional subshift $X^{\kappa}_{\rho,\eta}$ is not empty if and only if the one-dimensional subshift $X_{\delta^{\kappa}}$ is not empty.*

For $(\mathcal{A}, \rho, \eta, \Sigma^{\rho}, \Sigma^{\eta}, \kappa)$, we will have a C^*-symbolic dynamical system $(\mathcal{A}, \delta^{\kappa}, \Sigma_{\kappa})$ in Section 4. It presents the subshift $X_{\delta^{\kappa}}$. Since a subshift presented by a C^*-symbolic dynamical system is always not empty, one sees

Proposition 3.5. *The two-dimensional subshift $X^{\kappa}_{\rho,\eta}$ is not empty.*

4. C^*-textile dynamical systems and their C^*-algebras

The C^*-algbebra $\mathcal{O}^{\kappa}_{\rho,\eta}$ is defined to be the universal C^*-algebra $C^*(x, S_\alpha, T_a; x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta)$ generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma^\rho$, $T_a, a \in \Sigma^\eta$ subject to the following relations called $(\rho, \eta; \kappa)$:

$$\sum_{\beta\in\Sigma^\rho} S_\beta S_\beta^* = 1, \qquad xS_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \qquad S_\alpha^* x S_\alpha = \rho_\alpha(x), \tag{4.1}$$

$$\sum_{b\in\Sigma^\eta} T_b T_b^* = 1, \qquad xT_a T_a^* = T_a T_a^* x, \qquad T_a^* x T_a = \eta_a(x), \tag{4.2}$$

$$S_\alpha T_b = T_a S_\beta \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta) \tag{4.3}$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$. We will study the algebra $\mathcal{O}^{\kappa}_{\rho,\eta}$. If $\kappa(\alpha, b) = (a, \beta)$, we write as $(\alpha, b) \overset{\kappa}{\cong} (a, \beta)$.

Lemma 4.1. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, one has $T_a^* S_\alpha \neq 0$ if and only if there exist $b \in \Sigma^\eta, \beta \in \Sigma^\rho$ such that $(\alpha, b) \overset{\kappa}{\cong} (a, \beta)$.*

Proof. Suppose that $T_a^* S_\alpha \neq 0$. As $T_a^* S_\alpha = \sum_{b'\in\Sigma^\eta} T_a^* S_\alpha T_{b'} T_{b'}^*$, there exists $b' \in \Sigma^\eta$ such that $T_a^* S_\alpha T_{b'} \neq 0$. Hence $\eta_{b'} \circ \rho_\alpha \neq 0$ so that $(\alpha, b') \in \Sigma^{\rho\eta}$. Then one may find $(a', \beta') \in \Sigma^\rho$ such that $(\alpha, b') \overset{\kappa}{\cong} (a', \beta')$ and hence $S_\alpha T_{b'} = T_{a'} S_{\beta'}$. Since $0 \neq T_a^* S_\alpha T_{b'} = T_a^* T_{a'} S_{\beta'}$, one sees that $a = a'$. Putting $b = b', \beta = \beta'$, we have $\kappa(\alpha, b) = (a, \beta)$.

Suppose next that $\kappa(\alpha, b) = (a, \beta)$. Since $\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \neq 0$, one has $0 \neq S_\alpha T_b = T_a S_\beta$. It follows that $S_\beta^* T_a^* S_\alpha T_b = (T_a S_\beta)^* T_a S_\beta$ so that $T_a^* S_\alpha \neq 0$. $\square$

Lemma 4.2. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have*

$$T_a^* S_\alpha = \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\beta \eta_b(\rho_\alpha(1)) T_b^* \tag{4.4}$$

and hence

$$S_\alpha^* T_a = \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} T_b \rho_\beta(\eta_a(1)) S_\beta^*. \tag{4.5}$$

Proof. We may assume that $T_a^* S_\alpha \neq 0$. One has $T_a^* S_\alpha = \sum_{b'\in\Sigma^\eta} T_a^* S_\alpha T_{b'} T_{b'}^*$. For $b' \in \Sigma^\eta$ with $(\alpha, b') \in \Sigma^{\rho\eta}$, take $(a', \beta') \in \Sigma^{\eta\rho}$ such that $\kappa(\alpha, b') = (a', \beta')$ so that

$$T_a^* S_\alpha T_{b'} T_{b'}^* = T_a^* T_{a'} S_{\beta'} T_{b'}^*.$$

Hence $T_a^* S_\alpha T_{b'} T_{b'}^* \neq 0$ implies $a = a'$. Since $T_a^* T_a = \eta_a(1)$ which commutes with $S_{\beta'} S_{\beta'}^*$, we have

$$T_a^* T_a S_{\beta'} T_{b'}^* = S_{\beta'} S_{\beta'}^* T_a^* T_a S_{\beta'} T_{b'}^* = S_{\beta'} \rho_{\beta'}(\eta_a(1)) T_{b'}^* = S_{\beta'} \eta_{b'}(\rho_\alpha(1)) T_{b'}^*.$$

It follows that

$$T_a^* S_\alpha = \sum_{\substack{b',\beta' \\ \kappa(\alpha,b')=(a,\beta')}} T_a^* T_a S_{\beta'} T_{b'}^* = \sum_{\substack{b',\beta' \\ \kappa(\alpha,b')=(a,\beta')}} S_{\beta'} \eta_{b'}(\rho_\alpha(1)) T_{b'}^*.$$

$\square$

Hence we have

Lemma 4.3. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have*

$$T_a T_a^* S_\alpha S_\alpha^* = \sum_{\substack{b \\ \kappa(\alpha,b)=(a,\beta) \text{ for some } \beta}} S_\alpha T_b T_b^* S_\alpha^*. \tag{4.6}$$

Hence $T_a T_a^*$ commutes with $S_\alpha S_\alpha^*$.

Proof. By the preceding lemma, we have

$$\begin{aligned}
T_a T_a^* S_\alpha S_\alpha^* &= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} T_a S_\beta \eta_b(\rho_\alpha(1)) T_b^* S_\alpha^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha T_b \eta_b(\rho_\alpha(1)) T_b^* S_\alpha^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha \rho_\alpha(1)) T_b T_b^* S_\alpha^* \\
&= \sum_{\substack{b \\ \kappa(\alpha,b)=(a,\beta) \text{for some } \beta}} S_\alpha T_b T_b^* S_\alpha^*.
\end{aligned}$$

□

We also have

Lemma 4.4. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $x, y \in Z_{\mathcal{A}}$, $T_a y T_a^*$ commutes with $S_\alpha x S_\alpha^*$.*

Proof. It follows that

$$\begin{aligned}
T_a y T_a^* S_\alpha x S_\alpha^* &= T_a y \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\beta \eta_b(\rho_\alpha(1)) T_b^* x S_\alpha^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} T_a S_\beta S_\beta^* y S_\beta \eta_b(\rho_\alpha(1)) T_b^* x T_b T_b^* S_\alpha^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha T_b \rho_\beta(y) \eta_b(\rho_\alpha(1)) \eta_b(x) S_\beta^* T_a^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha T_b \eta_b(x) \eta_b(\rho_\alpha(1)) \rho_\beta(y) S_\beta^* T_a^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha x \rho_\alpha(1) T_b S_\beta^* y T_a^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha x S_\alpha^* S_\alpha T_b S_\beta^* T_a^* T_a y T_a^* \\
&= \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha x (S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a) y T_a^*.
\end{aligned}$$

Now if $(\alpha, b') \notin \Sigma^{\rho,\eta}$, then $S_\alpha T_{b'} = 0$. Hence

$$\sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a = \sum_b S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a = S_\alpha^* T_a.$$

Therefore we have

$$T_a y T_a^* S_\alpha x S_\alpha^* = S_\alpha x S_\alpha^* T_a y T_a^*.$$

□

For a subset F of $\mathcal{O}^\kappa_{\rho,\eta}$, denote by $C^*(F)$ the C^*-subalgebra of $\mathcal{O}^\kappa_{\rho,\eta}$ generated by the elements of F. We set

$$\begin{aligned}
\mathcal{D}_{\rho,\eta} =& C^*(S_\mu T_\zeta x T_\zeta^* S_\mu^* : \mu \in B_*(\Lambda_\rho), \zeta \in B_*(\Lambda_\eta), x \in \mathcal{A}),\\
\mathcal{D}_{j,k} =& C^*(S_\mu T_\zeta x T_\zeta^* S_\mu^* : \mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta), x \in \mathcal{A}) \text{ for } j,k \in \mathbb{Z}_+.
\end{aligned}$$

By the commutation relation (4.3), one sees that

$$\mathcal{D}_{j,k} = C^*(T_\xi S_\nu x S_\nu^* T_\xi^* : \nu \in B_j(\Lambda_\rho), \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}).$$

The identities

$$\begin{aligned}
S_\mu T_\zeta x T_\zeta^* S_\mu^* &= \sum_{a \in \Sigma^\eta} S_\mu T_{\zeta a} \eta_a(x) T_{\zeta a}^* S_\mu^*,\\
T_\xi S_\nu x S_\nu^* T_\xi^* &= \sum_{\alpha \in \Sigma^\rho} T_\xi S_{\nu\alpha} \rho_\alpha(x) S_{\nu\alpha}^* T_\xi^*
\end{aligned}$$

for $x \in \mathcal{A}$ and $\mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta)$ yield the embeddings

$$\mathcal{D}_{j,k} \hookrightarrow \mathcal{D}_{j,k+1}, \qquad \mathcal{D}_{j,k} \hookrightarrow \mathcal{D}_{j+1,k}$$

respectively such that $\cup_{j,k\in\mathbb{Z}_+} \mathcal{D}_{j,k}$ is dense in $\mathcal{D}_{\rho,\eta}$.

Proposition 4.5. *If $\mathcal{A}$ is commutative, so is $\mathcal{D}_{\rho,\eta}$.*

Proof. The preceding lemma tells us that $\mathcal{D}_{1,1}$ is commutative. Suppose that the algebra $\mathcal{D}_{j,k}$ is commutative for fixed $j,k \in \mathbb{N}$. We will show that the both algebras $\mathcal{D}_{j+1,k}$ and $\mathcal{D}_{j,k+1}$ are commutative. For the algebra $\mathcal{D}_{j+1,k}$, it consists of the linear span of elements of the form:

$$S_\alpha x S_\alpha^* \qquad \text{for } x \in \mathcal{D}_{j,k}, \alpha \in \Sigma^\rho.$$

For $x, y \in \mathcal{D}_{j,k}, \alpha, \beta \in \Sigma^\rho$, we will show that $S_\alpha x S_\alpha^*$ commutes with both $S_\beta y S_\beta^*$ and y. If $\alpha = \beta$, it is easy to see that $S_\alpha x S_\alpha^*$ commutes with $S_\alpha y S_\alpha^*$, because $\rho_\alpha(1) \in \mathcal{A} \subset \mathcal{D}_{j,k}$. If $\alpha \neq \beta$, both $S_\alpha x S_\alpha^* S_\beta y S_\beta^*$ and $S_\beta y S_\beta^* S_\alpha x S_\alpha^*$ are zeros. Since $S_\alpha^* y S_\alpha \in \mathcal{D}_{j-1,k} \subset \mathcal{D}_{j,k}$, one sees $S_\alpha^* y S_\alpha$ commutes with x. One also sees that $S_\alpha S_\alpha^* \in \mathcal{D}_{j,k}$ commutes with y. It follows that

$$S_\alpha x S_\alpha^* y = S_\alpha x S_\alpha^* y S_\alpha S_\alpha^* = S_\alpha S_\alpha^* y S_\alpha x S_\alpha^* = y S_\alpha x S_\alpha^*.$$

Hence the algebra $\mathcal{D}_{j+1,k}$ is commutative, and similarly so is $\mathcal{D}_{j,k+1}$. By induction, the algebras $\mathcal{D}_{j,k}$ are all commutative for all $j,k \in \mathbb{N}$. Since $\cup_{j,k\in\mathbb{N}} \mathcal{D}_{j,k}$ is dense in $\mathcal{D}_{\rho,\eta}$, $\mathcal{D}_{\rho,\eta}$ is commuatative. □

Proposition 4.6. *Let $\mathcal{O}^{alg}_{\rho,\eta}$ be the dense $*$-subalgebra algebraically generated by elements $x \in \mathcal{A}$, $S_\alpha, \alpha \in \Sigma^\rho$ and $T_a, a \in \Sigma^\eta$. Then each element of $\mathcal{O}^{alg}_{\rho,\eta}$ is a finite linear combination of elements of the form:*

$$S_\mu T_\zeta x T_\xi^* S_\nu^* \qquad \text{for } x \in \mathcal{A}, \mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta) \tag{4.7}$$

where $S_\mu = S_{\mu_1}\cdots S_{\mu_k}, S_\nu = S_{\nu_1}\cdots S_{\nu_n}$ *for* $\mu = \mu_1\cdots\mu_k, \nu = \nu_1\cdots\nu_n$ *and* $T_\zeta = T_{\zeta_1}\cdots T_{\zeta_l}, T_\xi = T_{\xi_1}\cdots T_{\xi_m}$ *for* $\zeta = \zeta_1\cdots\zeta_l, \xi = \xi_1\cdots\xi_m$.

Proof. For $\alpha,\beta\in\Sigma^\rho$, $a,b\in\Sigma^\eta$ and $x\in\mathcal{A}$, we have

$$S_\alpha^* S_\beta = \begin{cases}\rho_\alpha(1)\in\mathcal{A} & \text{if } \alpha=\beta,\\ 0 & \text{otherwise,}\end{cases} \qquad T_a^* T_b = \begin{cases}\eta_a(1)\in\mathcal{A} & \text{if } a=b,\\ 0 & \text{otherwise,}\end{cases}$$

$$S_\alpha^* T_a = \sum_{\substack{b,\beta\\ \kappa(\alpha,b)=(a,\beta)}} T_b\rho_\beta(\eta_a(1))S_\beta^*, \qquad T_a^* S_\alpha = \sum_{\substack{b,\beta\\ \kappa(\alpha,b)=(a,\beta)}} S_\beta\eta_b(\rho_\alpha(1))T_b^*,$$

$$S_\alpha^* x = \rho_\alpha(x)S_\alpha, \qquad T_a^* x = \eta_a(x)T_a^*,$$

And also

$$S_\beta^* T_a^* = \begin{cases} T_b^* S_\alpha^* & \text{if } (a,\beta)\in\Sigma^{\eta\rho} \text{ and } (a,\beta)=\kappa(\alpha,b),\\ 0 & \text{if } (a,\beta)\notin\Sigma^{\eta\rho}.\end{cases}$$

Therefore we conclude that any element of $\mathcal{O}_{\rho,\eta}^{alg}$ is a finite linear combination of elements of the form of (4.6). □

Similarly we have

Proposition 4.7. *Each element of* $\mathcal{O}_{\rho,\eta}^{alg}$ *is a finite linear combination of elements of the form:*

$$T_\zeta S_\mu x S_\nu^* T_\xi^* \qquad \text{for } x\in\mathcal{A}, \mu,\nu\in B_*(\Lambda_\rho), \zeta,\xi\in B_*(\Lambda_\eta). \tag{4.8}$$

In the rest of this section, we will have a C^*-symbolic dynamical system $(\mathcal{A},\delta^\kappa,\Sigma_\kappa)$ from $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$, which presents the one-dimensional subshift X_{δ^κ} described in the previous section. For $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$, define an endomorphism δ_ω^κ on $\mathcal{A}$ for $\omega\in\Sigma_\kappa$ by setting

$$\delta_\omega^\kappa(x) = \eta_b(\rho_\alpha(x))(=\rho_\beta(\eta_a(x))), \qquad x\in\mathcal{A}, \quad \omega=(\alpha,b,a,\beta)\in\Sigma_\kappa.$$

Lemma 4.8. $(\mathcal{A},\delta^\kappa,\Sigma_\kappa)$ *is a* C^**-symbolic dynamical system that presents* X_{δ^κ}*.*

Proof. We will show that δ^κ is essential and faithful. Now both C^*-symbolic dynamical systems $(\mathcal{A},\eta,\Sigma^\eta)$ and $(\mathcal{A},\rho,\Sigma^\eta)$ are essential. Since $\rho_\alpha(Z_\mathcal{A})\subset Z_\mathcal{A}$ and $\eta_a(Z_\mathcal{A})\subset Z_\mathcal{A}$, it is clear that $\delta_\omega^\kappa(Z_\mathcal{A})\subset Z_\mathcal{A}$. By the inequalities

$$\sum_{\omega\in\Sigma_\kappa}\delta_\omega^\kappa(1) = \sum_{b\in\Sigma^\eta}\sum_{\alpha\in\Sigma^\rho}\eta_b(\rho_\alpha(1)) \ge \sum_{b\in\Sigma^\eta}\eta_b(1)\ge 1$$

$\{\delta^\kappa\}_{\omega\in\Sigma_\kappa}$ is essential. For any nonzero $x\in\mathcal{A}$, there exists $\alpha\in\Sigma^\rho$ such that $\rho_\alpha(x)\ne 0$ and there exists $b\in\Sigma^\eta$ such that $\eta_b(\rho_\alpha(x))\ne 0$. Hence δ^κ is faithful so that $(\mathcal{A},\delta^\kappa,\Sigma_\kappa)$ is a C^*-symbolic dynamical system. It is obvious that the subshift presented by $(\mathcal{A},\delta^\kappa,\Sigma_\kappa)$ is X_{δ^κ}. □

Put

$$\widehat{X}_{\rho,\eta}^\kappa = \{(\omega_{i,-j})_{(i,j)\in\mathbb{N}^2}\in\Sigma_\kappa^{\mathbb{N}^2} \mid (\omega_{i,j})_{(i,j)\in\mathbb{Z}^2}\in X_{\rho,\eta}^\kappa\}$$

and

$$\widehat{X}_{\delta^\kappa} = \{(\omega_{n,-n})_{n\in\mathbb{N}}\in\Sigma_\kappa^{\mathbb{N}} \mid (\omega_{i,j})_{(i,j)\in\mathbb{N}^2}\in\widehat{X}_{\rho,\eta}^\kappa\}.$$

The latter set $\widehat{X}_{\delta^\kappa}$ is the right one-sided subshift for X_{δ^κ}.

Lemma 4.9. *A configuration* $(\omega_{i,-j})_{(i,j)\in\mathbb{N}^2} \in \widehat{X}^{\kappa}_{\rho,\eta}$ *can extend to a whole configuration* $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^{\kappa}_{\rho,\eta}$.

Proof. For $(\omega_{i,-j})_{(i,j)\in\mathbb{N}^2} \in \widehat{X}^{\kappa}_{\rho,\eta}$, put $x_i = \omega_{i,-i}, i \in \mathbb{N}$ so that $x = (x_i)_{i\in\mathbb{N}} \in \widehat{X}_{\delta^\kappa}$. Since $\widehat{X}_{\delta^\kappa}$ is a one-sided subshift, there exists an extension $\tilde{x} \in X_{\delta^\kappa}$ to two-sided sequence such that $\tilde{x}_{[1,\infty)} = x$. By the diagonal property, $\tilde{x}$ determines a whole configuration $\tilde{\omega}$ to $\mathbb{Z}^2$ such that $\tilde{\omega} \in X^{\kappa}_{\delta,\eta}$ and $(\tilde{\omega}_{i,-i})_{i\in\mathbb{N}} = \tilde{x}$. Hence $\tilde{\omega}_{i,-j} = \omega_{i,-j}$ for all $i, j \in \mathbb{N}$. □

Let $\mathfrak{D}_{\rho,\eta}$ be the C^*-subalgebra of $\mathcal{D}_{\rho,\eta}$ defined by

$$\begin{aligned}\mathfrak{D}_{\rho,\eta} &= C^*(S_\mu T_\zeta T^*_\zeta S^*_\mu : \mu \in B_*(\Lambda_\rho), \zeta \in B_*(\Lambda_\eta)) \\ &= C^*(T_\xi S_\nu S^*_\nu T^*_\xi : \nu \in B_*(\Lambda_\rho), \xi \in B_*(\Lambda_\eta))\end{aligned}$$

which is a commutative C^*-subalgebra of $\mathcal{D}_{\rho,\eta}$. Put for $\mu = \mu_1 \cdots \mu_n \in B_*(\Lambda_\rho)$, $\zeta = \zeta_1 \cdots \zeta_m \in B_*(\Lambda_\eta)$ the cylinder set

$$U_{\mu,\zeta} = \{(\omega_{i,-j})_{(i,j)\in\mathbb{N}^2} \in \widehat{X}^{\kappa}_{\rho,\eta} \mid t(\omega_{i,-1}) = \mu_i, i = 1, \cdots, n, r(\omega_{n,-j}) = \zeta_j, j = 1, \cdots, m\}$$

The following lemma is direct.

Lemma 4.10. $\mathfrak{D}_{\rho,\eta}$ *is isomorphic to* $C(\widehat{X}^{\kappa}_{\rho,\eta})$ *through the correspondence such that* $S_\mu T_\zeta T^*_\zeta S^*_\mu$ *sends to* $\chi_{U_{\mu,\zeta}}$, *where* $\chi_{U_{\mu,\zeta}}$ *is the characteristic function for the cylinder set* $U_{\mu,\zeta}$ *on* $\widehat{X}^{\kappa}_{\rho,\eta}$.

5. Condition (I) for C^*-textile dynamical systems

The notion of condition (I) for finite square matrices with entries in $\{0, 1\}$ has been introduced in [7]. The condition has been generalized by many authors to corresponding conditions for generalizations of the Cuntz–Krieger algebras (cf. [10], [12], [16], [36]). The condition (I) for C^*-symbolic dynamical systems (including λ-graph systems) has been also defined in [25](cf. [21], [22]). All of these conditions give rise to the uniqueness of the associated C^*-algebras subject to some operator relations among certain generating elements.

In this section, we will introduce the notion of condition (I) for C^*-textile dynamical systems to prove the uniqueness of the C^*-algebras $\mathcal{O}^{\kappa}_{\rho,\eta}$ under the relation $(\rho, \eta; \kappa)$.

Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-symbolic dynamical system over Σ and $X^{\kappa}_{\rho,\eta}$ the associated two-dimensional subshift. Denote by $\Lambda_\rho, \Lambda_\eta$ the associated subshifts to the C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho), (\mathcal{A}, \eta, \Sigma^\eta)$ respectively. For $\mu = (\mu_1, \dots, \mu_j) \in B_j(\Lambda_\rho), \zeta = (\zeta_1, \dots, \zeta_k) \in B_k(\Lambda_\eta)$, we put $\rho_\mu = \rho_{\mu_j} \circ \cdots \circ \rho_{\mu_1}, \eta_\mu = \eta_{\zeta_k} \circ \cdots \circ \eta_{\zeta_1}$ respectively. We denote by $|\mu|, |\zeta|$ the lengths j, k respectively. In the algebra $\mathcal{O}^{\kappa}_{\rho,\eta}$, we set the subalgebras

$$\mathcal{F}_{\rho,\eta} = C^*(S_\mu T_\zeta x T^*_\xi S^*_\nu : \mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta), |\mu| = |\nu|, |\zeta| = |\xi|, x \in \mathcal{A})$$

and for $j, k \in \mathbb{Z}_+$

$$\mathcal{F}_{j,k} = C^*(S_\mu T_\zeta x T^*_\xi S^*_\nu : \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}).$$

We notice that

$$\mathcal{F}_{j,k} = C^*(T_\zeta S_\mu x S^*_\nu T^*_\xi : \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}).$$

The identities

$$S_\mu T_\zeta x T_\xi^* S_\nu^* = \sum_{a\in\Sigma^\eta} S_\mu T_{\zeta a}\eta_a(x) T_{\xi a}^* S_\nu^*, \tag{5.1}$$

$$T_\zeta S_\mu x S_\nu^* T_\xi^* = \sum_{\alpha\in\Sigma^\rho} T_\zeta S_{\mu\alpha}\rho_\alpha(x) S_{\nu\alpha}^* T_\xi^* \tag{5.2}$$

for $x \in \mathcal{A}$ and $\mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta)$ yield the embeddings

$$\iota_{*,+1} : \mathcal{F}_{j,k} \hookrightarrow \mathcal{F}_{j,k+1}, \qquad \iota_{+1,*} : \mathcal{F}_{j,k} \hookrightarrow \mathcal{F}_{j+1,k}$$

such that $\cup_{j,k\in\mathbb{Z}_+}\mathcal{F}_{j,k}$ is dense in $\mathcal{F}_{\rho,\eta}$.

By the universality of $\mathcal{O}^\kappa_{\rho,\eta}$, we may define an action $\theta : \mathbb{T}^2 \longrightarrow \mathrm{Aut}(\mathcal{O}^\kappa_{\rho,\eta})$ of the 2-dimensional torus group $\mathbb{T}^2 = \{(z,w) \in \mathbb{C}^2 \mid |z| = |w| = 1\}$ to $\mathcal{O}^\kappa_{\rho,\eta}$ by setting

$$\theta_{z,w}(S_\alpha) = zS_\alpha, \quad \theta_{z,w}(T_a) = wT_a, \quad \theta_{z,w}(x) = x$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, $x \in \mathcal{A}$ and $z, w \in \mathbb{T}$. We call the action $\theta : \mathbb{T}^2 \longrightarrow \mathrm{Aut}(\mathcal{O}^\kappa_{\rho,\eta})$ the gauge action of $\mathbb{T}^2$ on $\mathcal{O}^\kappa_{\rho,\eta}$. The fixed point algebra of $\mathcal{O}^\kappa_{\rho,\eta}$ under θ is denoted by $(\mathcal{O}^\kappa_{\rho,\eta})^\theta$. Let $\mathcal{E}_{\rho,\eta} : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow (\mathcal{O}^\kappa_{\rho,\eta})^\theta$ be the conditional expectaton defined by

$$\mathcal{E}_{\rho,\eta}(X) = \int_{(z,w)\in\mathbb{T}^2} \theta_{z,w}(X)\, dzdw, \qquad X \in \mathcal{O}^\kappa_{\rho,\eta}.$$

The following lemma is routine.

Lemma 5.1. $(\mathcal{O}^\kappa_{\rho,\eta})^\theta = \mathcal{F}_{\rho,\eta}$.

Put $\phi_\rho, \phi_\eta : \mathcal{D}_{\rho,\eta} \longrightarrow \mathcal{D}_{\rho,\eta}$ by setting

$$\phi_\rho(X) = \sum_{\alpha\in\Sigma^\rho} S_\alpha X S_\alpha^*, \qquad \phi_\eta(X) = \sum_{a\in\Sigma^\eta} T_a X T_a^*, \qquad X \in \mathcal{D}_{\rho,\eta}.$$

It is easy to see by (4.3)

$$\phi_\rho \circ \phi_\eta = \phi_\eta \circ \phi_\rho \quad \text{ on } \mathcal{D}_{\rho,\eta}.$$

Definition. A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to satisfy *condition* (I) if there exists a unital increasing sequence $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$ of C^*-subalgebras of $\mathcal{A}$ such that

(1) $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$, $\eta_a(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbb{Z}_+, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$,

(2) $\cup_{l\in\mathbb{Z}_+}\mathcal{A}_l$ is dense in $\mathcal{A}$,

(3) for $\epsilon > 0$, $j, k, l \in \mathbb{N}$ with $j + k \le l$ and $X_0 \in \mathcal{F}^l_{j,k} = C^*(S_\mu T_\zeta x T_\xi^* S_\nu^* : \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}_l)$, there exists an element $g \in \mathcal{D}_{\rho,\eta} \cap {\mathcal{A}_l}'(= \{y \in \mathcal{D}_{\rho,\eta} \mid ya = ay \ \text{ for } a \in \mathcal{A}_l\})$ with $0 \le g \le 1$ such that

- (i) $\|X_0 \phi^j_\rho \circ \phi^k_\eta(g)\| \ge \|X_0\| - \epsilon$,
- (ii) $\phi^n_\rho(g)\phi^m_\eta(g) = \phi^n_\rho((\phi^m_\eta(g)))g = \phi^n_\rho(g)g = \phi^m_\eta(g)g = 0$ for all $n = 1, 2, \ldots, j$, $m = 1, 2, \ldots, k$.

If in particular, one may take the above subalgebras $\mathcal{A}_l \subset \mathcal{A}$, $l = 0, 1, 2, \ldots$ to be of finite dimensional, then $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to satisfy *AF-condition* (I). In this case, $\mathcal{A} = \overline{\cup_{l=0}^\infty \mathcal{A}_l}$ is an AF-algebra.

As the element g above belongs to the diagonal subalgebra $\mathcal{D}_{\rho,\eta}$ of $\mathcal{F}_{\rho,\eta}$, the condition (I) of $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is intrinsically determined by itself by virtue of Lemma 5.3 below.

We will also introduce the following condition called *free*, which will be stronger than condition (I) but easier to confirm than condition (I).

Definition. A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to be *free* if there exists a unital increasing sequence $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$ of C^*-subalgebras of $\mathcal{A}$ such that

(1) $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$, $\eta_a(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbb{Z}_+, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$,

(2) $\cup_{l \in \mathbb{Z}_+} \mathcal{A}_l$ is dense in $\mathcal{A}$,

(3) for $j, k, l \in \mathbb{N}$ with $j + k \le l$ there exists a projection $q \in \mathcal{D}_{\rho,\eta} \cap {\mathcal{A}_l}'$ such that

  (i) $qa \ne 0$ for $0 \ne a \in \mathcal{A}_l$,

  (ii) $\phi_\rho^n(q)\phi_\eta^m(q) = \phi_\rho^n((\phi_\eta^m(q)))q = \phi_\rho^n(q)q = \phi_\eta^m(q)q = 0$ for all $n = 1, 2, \dots, j$, $m = 1, 2, \dots, k$.

If in particular, one may take the above subalgebras $\mathcal{A}_l \subset \mathcal{A}$, $l = 0, 1, 2, \dots$ to be of finite dimensional, then $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to be *AF-free* .

Proposition 5.2. *If a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is free (resp. AF-free), then it satisfies condition (I) (resp. AF-condition (I)).*

Proof. Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is free. Take an increasing sequence $\mathcal{A}_l, l \in \mathbb{N}$ of C^*-subalgebras of $\mathcal{A}$ satisfying the above conditions (1),(2),(3) of freeness. For $j, k, l \in \mathbb{N}$ with $j + k \le l$ there exists a projection $q \in \mathcal{D}_{\rho,\eta} \cap {\mathcal{A}_l}'$ satisfying the above two conditions (i) and (ii) of (3). Put $Q^l_{j,k} = \phi_\rho^j(\phi_\eta^k(q))$. For $x \in \mathcal{A}_l, \mu, \nu \in B_j(\Lambda_\rho), \xi, \zeta \in B_k(\Lambda_\eta)$, one has the equality

$$Q^l_{j,k} S_\mu T_\zeta x T_\xi^* S_\nu^* = S_\mu T_\zeta x T_\xi^* S_\nu^*$$

so that $Q^l_{j,k}$ commutes with all of elements of $\mathcal{F}^l_{j,k}$. By using the condition (i) of (3) for q one directly sees that $S_\mu T_\zeta x T_\xi^* S_\nu^* \ne 0$ if and only if $Q^l_{j,k} S_\mu T_\zeta x T_\xi^* S_\nu^* \ne 0$. Hence the map

$$X \in \mathcal{F}^l_{j,k} \longrightarrow X Q^l_{j,k} \in \mathcal{F}^l_{j,k} Q^l_{j,k}$$

defines a homomorphism, that is proved to be injective by a similar proof to the proof of [26, Proposition 3.7]. Hence we have $\|X Q^l_{j,k}\| = \|X\| \ge \|X\| - \epsilon$ for all $X \in \mathcal{F}^l_{j,k}$. □

Let $\mathcal{B}$ be a unital C^*-algebra. Suppose that there exist an injective $*$-homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ preserving their units and two families $s_\alpha \in \mathcal{B}, \alpha \in \Sigma^\rho$ and $t_a \in \mathcal{B}, a \in \Sigma^\eta$ of partial isometries satisfying

$$\sum_{\beta \in \Sigma^\rho} s_\beta s_\beta^* = 1, \qquad \pi(x) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi(x), \qquad s_\alpha^* \pi(x) s_\alpha = \pi(\rho_\alpha(x)),$$

$$\sum_{b \in \Sigma^\eta} t_b t_b^* = 1, \qquad \pi(x) t_a t_a^* = t_a t_a^* \pi(x), \qquad t_a^* \pi(x) t_a = \pi(\eta_a(x)),$$

$$s_\alpha t_b = t_a s_\beta \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$. Put $\widetilde{\mathcal{A}} = \pi(\mathcal{A})$ and $\tilde{\rho}_\alpha(\pi(x)) = \pi(\rho_\alpha(x)), \tilde{\eta}_a(\pi(x)) = \pi(\eta_a(x)), x \in \mathcal{A}$. It is easy to see that $(\widetilde{\mathcal{A}}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \kappa)$ is a C^*-textile dynamical system such that the presented textile dynamical system $X^\kappa_{\tilde{\rho},\tilde{\eta}}$ is the same as the one $X^\kappa_{\rho,\eta}$ presented by $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$. Let $\mathcal{O}_{\pi,s,t}$ be the C^*-subalgebra of $\mathcal{B}$ generated by $\pi(x)$ and s_α, t_a for $x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$. Let $\mathcal{F}_{\pi,s,t}$ be the C^*-subalgebra of $\mathcal{O}_{\pi,s,t}$ generated by $s_\mu t_\zeta \pi(x) t_\xi^* s_\nu^*$ for $x \in \mathcal{A}$ and $\mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta)$ with $|\mu| = |\nu|, |\zeta| = |\xi|$. By the universality of the algebra $\mathcal{O}^\kappa_{\rho,\eta}$, the correspondence

$$x \in \mathcal{A} \longrightarrow \pi(x) \in \widetilde{A}, \qquad S_\alpha \longrightarrow s_\alpha, \quad \alpha \in \Sigma^\rho, \qquad T_a \longrightarrow t_a, \quad a \in \Sigma^\eta$$

extends to a surjective $*$-homomorphism $\tilde{\pi} : \mathcal{O}^{\kappa}_{\rho,\eta} \longrightarrow \mathcal{O}_{\pi,s,t}$.

Lemma 5.3. *The restriction of $\tilde{\pi}$ to the subalgebra $\mathcal{F}_{\rho,\eta}$ is a $*$-isomorphism from $\mathcal{F}_{\rho,\eta}$ to $\mathcal{F}_{\pi,s,t}$. Hence if $(\mathcal{A}, \rho, \eta, \Sigma^{\rho}, \Sigma^{\eta}, \kappa)$ satisfies condition (I) (resp. is free), $(\widetilde{\mathcal{A}}, \tilde{\rho}, \tilde{\eta}, \Sigma^{\rho}, \Sigma^{\eta}, \kappa)$ satisfies condition (I) (resp. is free).*

Proof. It suffices to show that $\tilde{\pi}$ is injective on $\mathcal{F}_{j,k}$ for all $j, k \in \mathbb{Z}$. Suppose

$$\sum_{\mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta)} s_\mu t_\zeta \pi(x_{\mu,\zeta,\xi,\nu}) t^*_\xi s^*_\nu = 0$$

for $\sum_{\mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta)} S_\mu T_\zeta x_{\mu,\zeta,\xi,\nu} T^*_\xi S^*_\nu \in \mathcal{F}_{j,k}$ with $x_{\mu,\zeta,\xi,\nu} \in \mathcal{A}$. For $\mu', \nu' \in B_j(\Lambda_\rho), \zeta', \xi' \in B_k(\Lambda_\eta)$, one has

$$\begin{aligned}
&\pi(\eta_{\zeta'}(\rho_{\mu'}(1)) x_{\mu',\zeta',\xi',\nu'} \eta_{\xi'}(\rho_{\nu'}(1))) \\
=& t^*_{\zeta'} s^*_{\mu'} \Big(\sum_{\mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta)} s_\mu t_\zeta \pi(x_{\mu,\zeta,\xi,\nu}) t^*_\xi s^*_\nu \Big) s_{\nu'} t_{\xi'} = 0.
\end{aligned}$$

As $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ is injective, one sees

$$\eta_{\zeta'}(\rho_{\mu'}(1)) x_{\mu',\zeta',\xi',\nu'} \eta_{\xi'}(\rho_{\nu'}(1)) = 0$$

so that

$$S_{\mu'} T_{\zeta'} x_{\mu',\zeta',\xi',\nu'} T^*_{\xi'} S^*_{\nu'} = 0.$$

Hence we have

$$\sum_{\mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta)} S_\mu T_\zeta x_{\mu,\zeta,\xi,\nu} T^*_\xi S^*_\nu = 0.$$

Therefore $\tilde{\pi}$ is injective on $\mathcal{F}_{j,k}$. $\square$

We henceforth assume that $(\mathcal{A}, \rho, \eta, \Sigma^{\rho}, \Sigma^{\eta}, \kappa)$ satisfies condition (I) defined above. Take a unital increasing sequence $\{\mathcal{A}_l\}_{l \in \mathbb{Z}_+}$ of C^*-subalgebras of $\mathcal{A}$ as in the definition of condition (I). Recall that the algebra $\mathcal{F}^l_{j,k}$ for $j, k \le l$ is defined as

$$\mathcal{F}^l_{j,k} = C^*(S_\mu T_\zeta x T^*_\xi S^*_\nu : \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}_l).$$

There exists an inclusion relation $\mathcal{F}^l_{j,k} \subset \mathcal{F}^{l'}_{j',k'}$ for $j \le j', k \le k'$ and $l \le l'$ through the identities (5.1), (5.2). Let $\mathcal{P}_{\pi,s,t}$ be the $*$-subalgebra of $\mathcal{O}_{\pi,s,t}$ algebraically generated by $\pi(x), s_\alpha, t_a$ for $x \in \mathcal{A}_l, l \in \mathbb{Z}_+,\ \alpha \in \Sigma^{\rho}, a \in \Sigma^{\eta}$.

Lemma 5.4. *Any element $x \in \mathcal{P}_{\pi,s,t}$ can be expressed in a unique way as*

$$\begin{aligned}
x =& \sum_{|\nu|,|\xi| \ge 1} x_{-\xi,-\nu} t^*_\xi s^*_\nu + \sum_{|\zeta|,|\nu| \ge 1} t_\zeta x_{\zeta,-\nu} s^*_\nu + \sum_{|\mu|,|\xi| \ge 1} s_\mu x_{\mu,-\xi} t^*_\xi + \sum_{|\mu|,\zeta| \ge 1} s_\mu t_\zeta x_{\mu,\zeta} \\
&+ \sum_{|\xi| \ge 1} x_{-\xi} t^*_\xi + \sum_{|\nu| \ge 1} x_{-\nu} s^*_\nu + \sum_{|\mu| \ge 1} s_\mu x_\mu + \sum_{|\zeta| \ge 1} t_\zeta x_\zeta + x_0
\end{aligned}$$

where the above summations Σ are all finite sums and the elements

$$x_{-\xi,-\nu}, x_{\zeta,-\nu}, x_{\mu,-\xi}, x_{\mu,\zeta}, x_{-\xi}, x_{-\nu}, x_\mu, x_\zeta, x_0 \text{ for } \mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta)$$

belong to the dense subalgebra $\mathcal{P}_{\pi,s,t} \cap \mathcal{F}_{\pi,s,t}$ which satisfy

$$\begin{aligned}
&x_{-\xi,-\nu} = x_{-\xi,-\nu} \eta_\xi(\rho_\nu(1)), \quad x_{\zeta,-\nu} = \eta_\zeta(1) x_{\zeta,-\nu} \rho_\nu(1), \\
&x_{\mu,-\xi} = \rho_\mu(1) x_{\mu,-\xi} \eta_\xi(1), \quad x_{\mu,\zeta} = \eta_\zeta(\rho_\mu(1)) x_{\mu,\zeta}, \\
&x_{-\xi} = x_{-\xi} \eta_\xi(1), \quad x_{-\nu} = x_{-\nu} \rho_\nu(1), \quad x_\mu = \rho_\mu(1) x_\mu, \quad x_\zeta = \eta_\zeta(1) x_\zeta.
\end{aligned}$$

Proof. Put

$$
\begin{aligned}
&x_{-\xi,-\nu} = \mathcal{E}_{\rho,\eta}(xs_\nu t_\xi), \quad x_{\zeta,-\nu} = \mathcal{E}_{\rho,\eta}(t_\zeta^* x s_\nu),\\
&x_{\mu,-\xi} = \mathcal{E}_{\rho,\eta}(s_\mu^* x t_\xi), \quad x_{\mu,\zeta} = \mathcal{E}_{\rho,\eta}(t_\zeta^* s_\mu^* x),\\
&x_{-\xi} = \mathcal{E}_{\rho,\eta}(xt_\xi), \quad x_{-\nu} = \mathcal{E}_{\rho,\eta}(xs_\nu), \quad x_\mu = \mathcal{E}_{\rho,\eta}(s_\mu^* x), \quad x_\zeta = \mathcal{E}_{\rho,\eta}(t_\zeta^* x),\\
&x_0 = \mathcal{E}_{\rho,\eta}(x).
\end{aligned}
$$

Then we have a desired expression of x. The elements

$$x_{-\xi,-\nu}, x_{\zeta,-\nu}, x_{\mu,-\xi}, x_{\mu,\zeta}, x_{-\xi}, x_{-\nu}, x_\mu, x_\zeta, x_0 \text{ for } \mu,\nu \in B_*(\Lambda_\rho), \zeta,\xi \in B_*(\Lambda_\eta)$$

are automatically determined by these formulae so that the unicity of the expression is clear. □

Lemma 5.5. *For $h \in \mathcal{D}_{\rho,\eta} \cap \mathcal{A}_l'$ and $j,k \in \mathbb{Z}$ with $j+k \le l$, put $h^{j,k} = \phi_\rho^j \circ \phi_\eta^k(h)$. Then we have*

(i) $h^{j,k}s_\mu = s_\mu h^{j-|\mu|,k}$ *for $\mu \in B_*(\Lambda_\rho)$ with $|\mu| \le j$.*
(ii) $h^{j,k}t_\zeta = t_\zeta h^{j,k-|\zeta|}$ *for $\zeta \in B_*(\Lambda_\eta)$ with $|\zeta| \le k$.*
(iii) $h^{j,k}$ *commutes with any element of $\mathcal{F}_{j,k}^l$.*

Proof. (i) It follows that for $\mu \in B_*(\Lambda_\rho)$ with $|\mu| \le j$

$$h^{j,k}s_\mu = \sum_{|\mu'|=|\mu|} s_{\mu'}\phi_\rho^{j-|\mu|}(\phi_\eta^k(h))s_{\mu'}^* s_\mu = s_\mu \phi_\rho^{j-|\mu|}(\phi_\eta^k(h))s_\mu^* s_\mu.$$

Since $h \in \mathcal{A}_l'$ and $\mathcal{A}_{j+k} \subset \mathcal{A}_l$, one has

$$
\begin{aligned}
\phi_\rho^{j-|\mu|}(\phi_\eta^k(h))s_\mu^* s_\mu &= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu t_\xi h t_\xi^* s_\nu^* s_\mu^* s_\mu\\
&= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu t_\xi h t_\xi^* s_\nu^* s_\mu^* s_\mu s_\nu t_\xi t_\xi^* s_\nu^*\\
&= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu t_\xi \eta_\xi(\rho_{\mu\nu}(1)) h t_\xi^* s_\nu^*\\
&= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_\nu \rho_{\mu\nu}(1) t_\xi h t_\xi^* s_\nu^*\\
&= s_\mu^* s_\mu \phi_\rho^{j-|\mu|}(\phi_\eta^k(h)) = s_\mu^* s_\mu h^{j-|\mu|,k}
\end{aligned}
$$

so that $h^{j,k}s_\mu = s_\mu h^{j-|\mu|,k}$.

(ii) Similarly we have $h^{j,k}t_\zeta = t_\zeta h^{j,k-|\zeta|}$ for $\zeta \in B_*(\Lambda_\eta)$ with $|\zeta| \le k$.

(iii) For $x \in \mathcal{A}_l, \mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta)$, we have

$$h^{j,k}s_\mu t_\zeta = s_\mu h^{0,k} t_\zeta = s_\mu t_\zeta h^{0,0} = s_\mu t_\zeta h.$$

It follows that

$$h^{j,k}s_\mu t_\zeta x t_\xi^* s_\nu^* = s_\mu t_\zeta h x t_\xi^* s_\nu^* = s_\mu t_\zeta x h t_\xi^* s_\nu^* = s_\mu t_\zeta x t_\xi^* s_\nu^* h^{j,k}.$$

Hence $h^{j,k}$ commutes with any element of $\mathcal{F}_{j,k}^l$. □

Lemma 5.6. *Assume that $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$ satisfies condition (I). Let $x \in \mathcal{P}_{\pi,s,t}$ be expressed as in the preceding lemma. Then we have*

$$\|x_0\| \le \|x\|.$$

Proof. We may assume that for $x \in \mathcal{P}_{\pi,s,t}$,

$$x_{-\xi,-\nu}, x_{\zeta,-\nu}, x_{\mu,-\xi}, x_{\mu,\zeta}, x_{-\xi}, x_{-\nu}, x_{\mu}, x_{\zeta}, x_0 \in \tilde{\pi}(\mathcal{F}^{l_1}_{j_1,k_1})$$

for some j_1, k_1, l_1 and $\mu, \nu \in \cup_{n=0}^{j_0} B_n(\Lambda_\rho)$, $\zeta, \xi \in \cup_{n=0}^{k_0} B_n(\Lambda_\eta)$ for some j_0, k_0. Take $j, k, l \in \mathbb{Z}_+$ such as

$$j \geq j_0 + j_1, \qquad k \geq k_0 + k_1, \qquad l \geq \max\{j+k, l_1\}.$$

By Lemma 5.3, $(\widetilde{\mathcal{A}}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). For any $\epsilon > 0$, the numbers j, k, l, and the element $x_0 \in \tilde{\pi}(\mathcal{F}^{l_1}_{j_1,k_1})$, one may find $g \in \tilde{\pi}(\mathcal{D}_{\rho,\eta}) \cap \pi(\mathcal{A}_l)'$ with $0 \leq g \leq 1$ such that

(i) $\|x_0 \phi^j_\rho \circ \phi^k_\eta(g)\| \geq \|x_0\| - \epsilon$,
(ii) $\phi^n_\rho(g)\phi^m_\eta(g) = \phi^n_\rho((\phi^m_\eta(g)))g = \phi^n_\rho(g)g = \phi^m_\eta(g)g = 0$ for all $n = 1, 2, \dots, j$, $m = 1, 2, \dots, k$.

Put $h = g^{\frac{1}{2}}$ and $h^{j,k} = \phi^j_\rho \circ \phi^k_\eta(h)$. It follows that

$$\begin{aligned}
\|x\| \geq {} & \|h^{j,k} x h^{j,k}\| \\
= {} & \| \sum_{|\nu|,|\xi| \geq 1} h^{j,k} x_{-\xi,-\nu} t^*_\xi s^*_\nu h^{j,k} \qquad (1) \\
& + \sum_{|\zeta|,|\nu| \geq 1} h^{j,k} t_\zeta x_{\zeta,-\nu} s^*_\nu h^{j,k} \qquad (2) \\
& + \sum_{|\mu|,|\xi| \geq 1} h^{j,k} s_\mu x_{\mu,-\xi} t^*_\xi h^{j,k} \qquad (3) \\
& + \sum_{|\mu|,\zeta| \geq 1} h^{j,k} s_\mu t_\zeta x_{\mu,\zeta} h^{j,k} \qquad (4) \\
& + \sum_{|\xi| \geq 1} h^{j,k} x_{-\xi} t^*_\xi h^{j,k} + \sum_{|\nu| \geq 1} h^{j,k} x_{-\nu} s^*_\nu h^{j,k} + \sum_{|\mu| \geq 1} h^{j,k} s_\mu x_\mu h^{j,k} + \sum_{|\zeta| \geq 1} h^{j,k} t_\zeta x_\zeta h^{j,k} \qquad (5) \\
& + h^{j,k} x_0 h^{j,k} \|
\end{aligned}$$

For (1), as $x_{-\xi,-\nu} \in \tilde{\pi}(\mathcal{F}^{l_1}_{j_1,k_1}) \subset \tilde{\pi}(\mathcal{F}^{l}_{j,k})$, one sees that $x_{-\xi,-\nu}$ commutes with $h^{j,k}$. Hence we have

$$h^{j,k} x_{-\xi,-\nu} t^*_\xi s^*_\nu h^{j,k} = x_{-\xi,-\nu} h^{j,k} t^*_\xi s^*_\nu h^{j,k} = x_{-\xi,-\nu} h^{j,k} h^{j-|\nu|,k-|\xi|} t^*_\xi s^*_\nu$$

and

$$\begin{aligned}
h^{j,k} h^{j-|\nu|,k-|\xi|} (h^{j,k} h^{j-|\nu|,k-|\xi|})^* = {} & \phi^j_\rho(\phi^k_\eta(g)) \cdot \phi^{j-|\nu|}_\rho(\phi^{k-|\xi|}_\eta(g)) \\
= {} & \phi^{j-|\nu|}_\rho \circ \phi^{k-|\xi|}_\eta(\phi^{|\xi|}_\eta(\phi^{|\nu|}_\rho(g)g)) = 0
\end{aligned}$$

so that

$$h^{j,k} x_{-\xi,-\nu} t^*_\xi s^*_\nu h^{j,k} = 0.$$

For (2), as $x_{\xi,-\nu} \in \tilde{\pi}(\mathcal{F}^{l_1}_{j_1,k_1}) \subset \tilde{\pi}(\mathcal{F}^{l}_{j,k-|\xi|})$, one sees that $x_{\xi,-\nu}$ commutes with $h^{j,k-|\xi|}$. Hence we have

$$h^{j,k} t_\xi x_{\xi,-\nu} s^*_\nu h^{j,k} = t_\xi h^{j,k-|\xi|} x_{\xi,-\nu} h^{j-|\nu|,k} s^*_\nu = t_\xi x_{\xi,-\nu} h^{j,k-|\xi|} h^{j-|\nu|,k} s^*_\nu$$

and

$$\begin{aligned}h^{j,k-|\xi|}h^{j-|\nu|,k}(h^{j,k-|\xi|}h^{j-|\nu|,k})^* =&\phi_\rho^j(\phi_\eta^{k-|\zeta|}(g))\cdot\phi_\rho^{j-|\nu|}(\phi_\eta^k(g))\\ =&\phi_\rho^{j-|\nu|}\circ\phi_\eta^{k-|\zeta|}(\phi_\rho^{|\nu|}(g)\phi_\eta^{|\zeta|}(g))=0\end{aligned}$$

so that

$$h^{j,k}t_\xi x_{\xi,-\nu}s_\nu^* h^{j,k}=0.$$

For (3), as $x_{\mu,-\xi}\in\tilde{\pi}(\mathcal{F}_{j_1,k_1}^{l_1})\subset\tilde{\pi}(\mathcal{F}_{j-|\mu|,k}^{l})$, one sees that $x_{\mu,-\xi}$ commutes with $h^{j-|\mu|,k}$. Hence we have

$$h^{j,k}s_\mu x_{\mu,-\xi}t_\xi^* h^{j,k}=s_\mu h^{j-|\mu|,k}x_{\mu,-\xi}h^{j,k-|\xi|}t_\xi^*=s_\mu x_{\mu,-\xi}h^{j-|\mu|,k}h^{j,k-|\xi|}t_\xi^*$$

and

$$\begin{aligned}h^{j-|\mu|,k}h^{j,k-|\xi|}(h^{j-|\mu|,k}h^{j,k-|\xi|})^* =&\phi_\rho^{j-|\mu|}(\phi_\eta^k(g))\cdot\phi_\rho^j(\phi_\eta^{k-|\xi|}(g))\\ =&\phi_\rho^{j-|\mu|}\circ\phi_\eta^{k-|\xi|}(\phi_\eta^{|\xi|}(g)\phi_\rho^{|\mu|}(g))=0\end{aligned}$$

so that

$$h^{j,k}s_\mu x_{\mu,-\xi}t_\xi^* h^{j,k}=0.$$

For (4), as $x_{\mu,\zeta}\in\tilde{\pi}(\mathcal{F}_{j_1,k_1}^{l_1})\subset\tilde{\pi}(\mathcal{F}_{j-|\mu|,k-|\zeta|}^{l})$, one sees that $x_{\mu,\zeta}$ commutes with $h^{j-|\mu|,k-|\zeta|}$. Hence we have

$$h^{j,k}s_\mu t_\zeta x_{\mu,\zeta}h^{j,k}=s_\mu t_\zeta h^{j-|\mu|,k-|\zeta|}x_{\mu,\zeta}h^{j,k}=s_\mu t_\zeta x_{\mu,\zeta}h^{j-|\mu|,k-|\zeta|}h^{j,k}$$

and

$$\begin{aligned}h^{j-|\mu|,k-|\zeta|}h^{j,k}(h^{j-|\mu|,k-|\zeta|}h^{j,k})^* =&\phi_\rho^{j-|\mu|}(\phi_\eta^{k-|\zeta|}(g))\phi_\rho^j(\phi_\eta^k(g))\\ =&\phi_\rho^{j-|\mu|}\circ\phi_\eta^{k-|\xi|}(g\phi_\rho^{|\mu|}\circ\phi_\eta^{|\xi|}(g))=0\end{aligned}$$

so that

$$h^{j,k}s_\mu t_\zeta x_{\mu,\zeta}h^{j,k}=0.$$

For (5) as $x_{-\xi}$ commutes with $h^{j,k}$, we have

$$h^{j,k}x_{-\xi}t_\xi^* h^{j,k}=x_{-\xi}h^{j,k}h^{j,k-|\xi|}t_\xi^*$$

and

$$\begin{aligned}h^{j,k}h^{j,k-|\xi|}(h^{j,k}h^{j,k-|\xi|})^* =&\phi_\rho^j(\phi_\eta^{k|}(g))\phi_\rho^j(\phi_\eta^{k-|\xi|}(g))\\ =&\phi_\rho^j\circ\phi_\eta^{k-|\xi|}(\phi_\eta^{|\xi|}(g))=0\end{aligned}$$

so that

$$h^{j,k}x_{-\xi}t_\xi^* h^{j,k}=0.$$

We similarly see that

$$h^{j,k}x_{-\nu}s_\nu^* h^{j,k}=h^{j,k}s_\mu x_\mu h^{j,k}=h^{j,k}t_\zeta x_\zeta h^{j,k}=0.$$

Therefore we have

$$\|x\|\geq\|h^{j,k}x_0h^{j,k}\|=\|x_0(h^{j,k})^2\|=\|x_0\phi_\rho^j\circ\phi_\eta^k(g)\|\geq\|x_0\|-\epsilon.$$

□

By a similar argument ot [7, 2.8 Proposition], one sees

Corollary 5.7. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). There exists a conditional expectation $\mathcal{E}_{\pi,s,t} : \mathcal{O}_{\pi,s,t} \longrightarrow \mathcal{F}_{\pi,s,t}$ such that $\mathcal{E}_{\pi,s,t} \circ \tilde{\pi} = \tilde{\pi} \circ \mathcal{E}_{\rho,\eta}$.*

Therefore we have

Proposition 5.8. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). The $*$-homomorphism $\tilde{\pi} : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow \mathcal{O}_{\pi,s,t}$ defined by*

$$\tilde{\pi}(x) = \pi(x), \quad x \in \mathcal{A}, \qquad \tilde{\pi}(S_\alpha) = s_\alpha, \quad \alpha \in \Sigma^\rho, \qquad \tilde{\pi}(T_a) = t_a, \quad a \in \Sigma^\eta$$

becomes a surjective $*$-isomorphism, and hence the C^*-algebras $\mathcal{O}^\kappa_{\rho,\eta}$ and $\mathcal{O}_{\pi,s,t}$ are canonically $*$-isomorphic through $\tilde{\pi}$.

Proof. The map $\tilde{\pi} : \mathcal{F}_{\rho,\eta} \to \mathcal{F}_{\pi,s,t}$ is $*$-isomorphic and satisfies $\mathcal{E}_{\pi,s,t} \circ \tilde{\pi} = \tilde{\pi} \circ \mathcal{E}_{\rho,\eta}$. Since $\mathcal{E}_{\rho,\eta} : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow \mathcal{F}_{\rho,\eta}$ is faithful, a routine argument shows that the $*$-homomorphism $\tilde{\pi} : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow \mathcal{O}_{\pi,s,t}$ is actually a $*$-isomorphism. $\square$

Hence the following uniqueness of the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ holds.

Theorem 5.9. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). The C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ is the unique C^*-algebra subject to the relation $(\rho, \eta; \kappa)$. This means that if there exist a unital C^*-algebra $\mathcal{B}$ and an injective $*$-homomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ and two families of partial isometries $s_\alpha, \alpha \in \Sigma^\rho$, $t_a, a \in \Sigma^\eta$ satisfying the following relations :*

$$\sum_{\beta \in \Sigma^\rho} s_\beta s_\beta^* = 1, \qquad \pi(x) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi(x), \qquad s_\alpha^* \pi(x) s_\alpha = \pi(\rho_\alpha(x)),$$
$$\sum_{b \in \Sigma^\eta} t_b t_b^* = 1, \qquad \pi(x) t_a t_a^* = t_a t_a^* \pi(x), \qquad t_a^* \pi(x) t_a = \pi(\eta_a(x))$$
$$s_\alpha t_b = t_a s_\beta \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta)$$

for $(\alpha, b) \in \Sigma^{\rho\eta}, (a, \beta) \in \Sigma^{\eta\rho}$ and $x \in \mathcal{A}$, $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, then the correspondence

$$x \in \mathcal{A} \longrightarrow \pi(x) \in \mathcal{B}, \quad S_\alpha \longrightarrow s_\alpha \in \mathcal{B}, \qquad T_a \longrightarrow t_a \in \mathcal{B}$$

extends to a $*$-isomorphism $\tilde{\pi}$ from $\mathcal{O}^\kappa_{\rho,\eta}$ onto the C^*-subalgebra $\mathcal{O}_{\pi,s,t}$ of $\mathcal{B}$ generated by $\pi(x), x \in \mathcal{A}$ and $s_\alpha, \alpha \in \Sigma$, $t_a, a \in \Sigma^\eta$.

For a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, let $\lambda_{\rho,\eta} : \mathcal{A} \to \mathcal{A}$ be the positive map on $\mathcal{A}$ defined by

$$\lambda_{\rho,\eta}(x) = \sum_{\alpha \in \Sigma^\rho, a \in \Sigma^\eta} \eta_a \circ \rho_\alpha(x), \qquad x \in \mathcal{A}.$$

Then $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to be *irreducible* if there exists no nontrivial ideal of $\mathcal{A}$ invariant under $\lambda_{\rho,\eta}$.

Corollary 5.10. *If $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I) and is irreducible, the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ is simple.*

Proof. Assume that there exists a nontrivial ideal $\mathcal{I}$ of $\mathcal{O}^\kappa_{\rho,\eta}$. Now suppose that $\mathcal{I} \cap \mathcal{A} = \{0\}$. As $S_\alpha^* S_\alpha = \rho_\alpha(1), T_a^* T_a = \eta_a(1) \in \mathcal{A}$ one knows that $S_\alpha, T_a \notin \mathcal{I}$ for all $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$. By the preceding theorem, the quotient map $q : \mathcal{O}^\kappa_{\rho,\eta} \longrightarrow \mathcal{O}^\kappa_{\rho,\eta}/\mathcal{I}$ must be injective so that $\mathcal{I}$ is trivial. Hence one sees that $\mathcal{I} \cap \mathcal{A} \neq \{0\}$ and it is invariant under $\lambda_{\rho,\eta}$. $\square$

6. Concrete realization

In this section we will realize the C^*-algebra $\mathcal{O}^{\kappa}_{\rho,\eta}$ for $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ in a concrete way as a C^*-algebra constructed from a Hilbert C^*-bimodule. For $\gamma_i \in \Sigma^\rho \cup \Sigma^\eta$, put

$$\xi_{\gamma_i} = \begin{cases} \rho_{\gamma_i} & \text{if } \gamma_i \in \Sigma^\rho, \\ \eta_{\gamma_i} & \text{if } \gamma_i \in \Sigma^\eta. \end{cases}$$

Definition. A finite sequence of labeles $(\gamma_1, \gamma_2, \dots, \gamma_k) \in (\Sigma^\rho \cup \Sigma^\eta)^k$ is said to be *concatenated labeled path* if $\xi_{\gamma_k} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1}(1) \neq 0$. For $m, n \in \mathbb{Z}_+$, let $L_{(n,m)}$ be the set of concatenated labeled paths $(\gamma_1, \gamma_2, \dots, \gamma_{m+n})$ such that symbols in Σ^ρ appear in $(\gamma_1, \gamma_2, \dots, \gamma_{m+n})$ n-times and symbols in Σ^η appear in $(\gamma_1, \gamma_2, \dots, \gamma_{m+n})$ m-times. We define a relation in $L_{(n,m)}$ for $i = 1, 2, \dots, n+m-1$. We write

$$(\gamma_1, \dots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{m+n}) \underset{i}{\approx} (\gamma_1, \dots, \gamma_{i-1}, \gamma'_i, \gamma'_{i+1}, \gamma_{i+2}, \dots, \gamma_{m+n})$$

if one of the following two conditions holds:
(1) $(\gamma_i, \gamma_{i+1}) \in \Sigma^{\rho\eta}, (\gamma'_i, \gamma'_{i+1}) \in \Sigma^{\eta\rho}$ and $\kappa(\gamma_i, \gamma_{i+1}) = (\gamma'_i, \gamma'_{i+1})$,
(2) $(\gamma_i, \gamma_{i+1}) \in \Sigma^{\eta\rho}, (\gamma'_i, \gamma'_{i+1}) \in \Sigma^{\rho\eta}$ and $\kappa(\gamma'_i, \gamma'_{i+1}) = (\gamma_i, \gamma_{i+1})$.

Denote by $\approx$ the equivalence relation in $L_{(n,m)}$ generated by the relations $\underset{i}{\approx}, i = 1, 2, \dots, n+m-1$. Let $\mathfrak{T}_{(n,m)} = L_{(n,m)}/\approx$ be the set of equivalence classes of $L_{(n,m)}$ under $\approx$. Denote by $[\gamma] \in \mathfrak{T}_{(n,m)}$ the equivalence class of $\gamma \in L_{(n,m)}$. Put the vectors $e = (1,0), f = (0,-1)$ in $\mathbb{R}^2$. Consider the set of all paths consisting of sequences of vectors e, f starting at the point $(-n, m) \in \mathbb{R}^2$ for $n, m \in \mathbb{Z}_+$ and ending at the origin. Such a path consists of n e-vectors and m f-vectors. Let $\mathfrak{P}_{(n,m)}$ be the set of all such paths from $(-n, m)$ to the origin. We consider the correspondence

$$\rho_\alpha \longrightarrow e \quad (\alpha \in \Sigma^\rho), \qquad \eta_a \longrightarrow f \quad (a \in \Sigma^\eta),$$

denoted by π. It extends a surjective map from $L_{(n,m)}$ to $\mathfrak{P}_{(n,m)}$ in a natural way. For a concatenated labeled path $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{n+m}) \in L_{(n,m)}$, put the projection in $\mathcal{A}$

$$P_\gamma = \xi_{\gamma_k} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1}(1).$$

We note that $P_\gamma \neq 0$ for all $\gamma \in L_{(n,m)}$.

Lemma 6.1. *For $\gamma, \gamma' \in L_{(n,m)}$, if $\gamma \approx \gamma'$, we have $P_\gamma = P_{\gamma'}$. Hence the projection $P_{[\gamma]}$ for $[\gamma] \in \mathfrak{T}_{(n,m)}$ is well-defined.*

Proof. If $\kappa(\alpha, b) = (a, \beta)$, one has $\eta_b \circ \rho_\alpha(1) = \rho_\beta \circ \eta_a(1) \neq 0$. Hence the assertion is obvious. $\square$

Denote by $|\mathfrak{T}_{(n,m)}|$ the cardinal number of the finite set $\mathfrak{T}_{(n,m)}$. Let $e_t, t \in \mathfrak{T}_{(n,m)}$ be the standard complete orthonomal basis of $\mathbb{C}^{|\mathfrak{T}_{(n,m)}|}$. Define

$$H_{(n,m)} = \sum_{t \in \mathfrak{T}_{(n,m)}} {}^{\oplus}\mathbb{C}e_t \otimes P_t\mathcal{A}$$
$$(= \sum_{t \in \mathfrak{T}_{(n,m)}} {}^{\oplus}\mathrm{Span}\{ce_t \otimes P_t x \mid c \in \mathbb{C}, x \in \mathcal{A}\})$$

the direct sum of $\mathbb{C}e_t \otimes P_t\mathcal{A}$ over $t \in \mathfrak{T}_{(n,m)}$. $H_{(n,m)}$ has a structure of C^*-bimodule over $\mathcal{A}$ by setting

$$(e_t \otimes P_t x)y := e_t \otimes P_t xy,$$
$$\phi(y)(e_t \otimes P_t x) := e_t \otimes \xi_\gamma(y)x (= e_t \otimes P_t \xi_\gamma(y)x) \quad \text{for } x, y \in \mathcal{A}$$

where $t = [\gamma]$ for $\gamma = (\gamma_1, \dots, \gamma_{n+m})$ and $\xi_\gamma(y) = \xi_{\gamma_{n+m}} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1}(y)$. Define an $\mathcal{A}$-valued inner product on $H_{(n,m)}$ by setting

$$\langle e_t \otimes P_t x \mid e_s \otimes P_s y \rangle := \begin{cases} x^* P_t y & \text{if } t = s, \\ 0 & \text{otherwise} \end{cases}$$

for $t, s \in \mathfrak{T}_{(n,m)}$ and $x, y \in \mathcal{A}$. Then $H_{(n,m)}$ becomes a Hilbert C^*-bimodule over $\mathcal{A}$. Put $H_{(0,0)} = \mathcal{A}$. Denote by F_κ the Hilbert C^*-bimodule over $\mathcal{A}$ defined by the direct sum:

$$F_\kappa = \sum_{(n,m)\in\mathbb{Z}_+^2} {}^{\oplus} H_{(n,m)}.$$

For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, the creation operators s_α, t_a on F_κ :

$$s_\alpha : H_{(n,m)} \longrightarrow H_{(n+1,m)}, \qquad t_a : H_{(n,m)} \longrightarrow H_{(n,m+1)}$$

are defined by

$$s_\alpha x = e_{[\alpha]} \otimes P_{[\alpha]} x, \qquad \text{for } x \in H_{(0,0)} (= \mathcal{A}),$$
$$s_\alpha(e_{[\gamma]} \otimes P_{[\gamma]} x) = \begin{cases} e_{[\alpha\gamma]} \otimes P_{[\alpha\gamma]} x & \text{if } \alpha\gamma \in L_{(n+1,m)}, \\ 0 & \text{otherwise,} \end{cases}$$
$$t_a x = e_{[a]} \otimes P_{[a]} x, \qquad \text{for } x \in H_{(0,0)} (= \mathcal{A}),$$
$$t_a(e_{[\gamma]} \otimes P_{[\gamma]} x) = \begin{cases} e_{[a\gamma]} \otimes P_{[a\gamma]} x & \text{if } a\gamma \in L_{(n,m+1)}, \\ 0 & \text{otherwise.} \end{cases}$$

For $y \in \mathcal{A}$ an operator $i_{F_\kappa}(y)$ on F_κ:

$$i_{F_\kappa}(y) : H_{(n,m)} \longrightarrow H_{(n,m)}$$

is defined by

$$i_{F_\kappa}(y)x = yx \qquad \text{for } x \in H_{(0,0)} (= \mathcal{A}),$$
$$i_{F_\kappa}(y)(e_{[\gamma]} \otimes P_{[\gamma]} x) = \phi(y)(e_{[\gamma]} \otimes P_{[\gamma]} x) (= e_{[\gamma]} \otimes \xi_\gamma(y)x).$$

Define the Cuntz–Toeplitz C^*-algebra for $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ by

$$\mathcal{T}_{\rho,\eta}^\kappa = C^*(s_\alpha, t_a, i_{F_\kappa}(y) \mid \alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A})$$

as the C^*-algebra on F_κ generated by $s_\alpha, t_a, i_{F_\kappa}(y)$ for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A}$.

Lemma 6.2. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have*

(i) $$s_\alpha^*(e_{[\gamma]} \otimes P_{[\gamma]} x) = \begin{cases} \phi(\rho_\alpha(1))(e_{[\gamma']} \otimes P_{[\gamma']} x) & \textit{if } \gamma \approx \alpha\gamma', \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) $$t_a^*(e_{[\gamma]} \otimes P_{[\gamma]} x) = \begin{cases} \phi(\eta_a(1))(e_{[\gamma']} \otimes P_{[\gamma']} x) & \textit{if } \gamma \approx a\gamma', \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. (i) For $\gamma \in L_{(n,m)}, \gamma' \in L_{(n-1,m)}$ and $\alpha \in \Sigma^\rho$, we have

$$\begin{aligned}\langle s_\alpha^*(e_{[\gamma]} \otimes P_{[\gamma]}x) \mid e_{[\gamma']} \otimes P_{[\gamma']}x' \rangle &= \langle e_{[\gamma]} \otimes P_{[\gamma]}x \mid e_{[\alpha\gamma']} \otimes P_{[\alpha\gamma']}x' \rangle \\ &= \begin{cases} x^* P_{[\alpha\gamma']}x & \text{if } \gamma \approx \alpha\gamma', \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

On the other hand,

$$\phi(\rho_\alpha(1))(e_{[\gamma']} \otimes P_{[\gamma']}x) = e_{[\gamma']} \otimes P_{[\alpha\gamma']}P_{\gamma'}x = e_{[\gamma']} \otimes P_{[\alpha\gamma']}x$$

so that

$$\langle \phi(\rho_\alpha(1))(e_{[\gamma']} \otimes P_{[\gamma']}x) \mid e_{[\gamma']} \otimes P_{[\gamma']}x' \rangle = x^* P_{[\alpha\gamma']}x'.$$

Hence we obtain the desired equality. Similarly we see (ii). □

The following lemma is straightforward.

Lemma 6.3. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $\gamma \in L_{(n,m)}, x \in \mathcal{A}$, we have*

(i)

$$s_\alpha s_\alpha^*(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} e_{[\gamma]} \otimes P_{[\gamma]}x) & \text{if } \gamma \approx \alpha\gamma' \text{ for some } \gamma' \in L_{(n-1,m)}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii)

$$t_a t_a^*(e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} e_{[\gamma]} \otimes P_{[\gamma]}x) & \text{if } \gamma \approx a\gamma' \text{ for some } \gamma' \in L_{(n,m-1)}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we see

Lemma 6.4.

(i) $1 - \sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^* =$ *the projection onto the subspace spanned by the vectors* $e_{[\gamma]} \otimes P_{[\gamma]}x$ *for* $\gamma \in \cup_{m=0}^\infty L_{(0,m)}, x \in \mathcal{A}$.
(ii) $1 - \sum_{a \in \Sigma^\eta} t_a t_a^* =$ *the projection onto the subspace spanned by the vectors* $e_{[\gamma]} \otimes P_{[\gamma]}x$ *for* $\gamma \in \cup_{n=0}^\infty L_{(n,0)}, x \in \mathcal{A}$.

Lemma 6.5. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $x \in \mathcal{A}$, we have*

(i) $s_\alpha^* x s_\alpha = \phi(\rho_\alpha(x))$ *and in particular* $s_\alpha^* s_\alpha = \phi(\rho_\alpha(1))$.
(ii) $t_a^* x t_a = \phi(\eta_a(x))$ *and in particular* $t_a^* t_a = \phi(\eta_a(1))$.

Proof. (i) It follows that for $\gamma \in L(n,m)$ with $\alpha\gamma \in L(n+1,m)$ and $y \in \mathcal{A}$,

$$\begin{aligned} s_\alpha^* x s_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}y) &= s_\alpha^*(e_{[\alpha\gamma]} \otimes P_{[\alpha\gamma]}y\xi_{\alpha\gamma}(x)) \\ &= e_{[\gamma]} \otimes P_{[\gamma]}y\xi_\gamma(\rho_\alpha(x))) \\ &= \phi(\rho_\alpha(x))(e_{[\gamma]} \otimes P_{[\gamma]}y). \end{aligned}$$

If $\alpha\gamma \notin L(n+1,m)$, we have

$$s_\alpha(e_{[\gamma]} \otimes P_{[\gamma]}y) = 0, \qquad \phi(\rho_\alpha(x))(e_{[\gamma]} \otimes P_{[\gamma]}y) = 0.$$

Hence we see that $s_\alpha^* x s_\alpha = \phi(\rho_\alpha(x))$.
(ii) The proof is similar to (i). □

Lemma 6.6. *For $\alpha, \beta \in \Sigma^\rho$, $a, b \in \Sigma^\eta$ we have*

$$s_\alpha t_b = t_a s_\beta \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{6.1}$$

Proof. For $\gamma \in L_{(n,m)}$ with $\alpha b\gamma, a\beta\gamma \in L_{(n+1,m+1)}$ and $x \in \mathcal{A}$, we have

$$
\begin{aligned}
s_\alpha t_b(e_{[\gamma]} \otimes P_{[\gamma]}x) &= e_{[\alpha b\gamma]} \otimes P_{[\alpha b\gamma]}y), \\
t_a s_\beta(e_{[\gamma]} \otimes P_{[\gamma]}x) &= (e_{[a\beta\gamma]} \otimes P_{[a\beta\gamma]}x).
\end{aligned}
$$

Since $\kappa(\alpha, b) = (a, \beta)$, the condition $\alpha b\gamma \in L_{(n+1,m+1)}$ is equivalent to the condition $a\beta\gamma \in L_{(n+1,m+1)}$. We then have $[\alpha b\gamma] = [a\beta\gamma]$ and $P_{[\alpha b\gamma]} = P_{[a\beta\gamma]}$. $\square$

Let $\mathcal{I}^\kappa_{\rho,\eta}$ be the ideal of $\mathcal{T}^\kappa_{\rho,\eta}$ generated by the two projections: $1 - \sum_{\alpha\in\Sigma^\rho} s_\alpha s_\alpha^*$ and $1 - \sum_{a\in\Sigma^\eta} t_a t_a^*$. Let $\widehat{\mathcal{O}}^\kappa_{\rho,\eta}$ be the quotient C^*-algebra

$$
\widehat{\mathcal{O}}^\kappa_{\rho,\eta} = \mathcal{T}^\kappa_{\rho,\eta} / \mathcal{I}^\kappa_{\rho,\eta}.
$$

Let $\pi_{\rho,\eta} : \mathcal{T}^\kappa_{\rho,\eta} \longrightarrow \widehat{\mathcal{O}}^\kappa_{\rho,\eta}$ be the quatient map. Put

$$
\widehat{S}_\alpha = \pi_{\rho,\eta}(s_\alpha), \quad \widehat{T}_a = \pi_{\rho,\eta}(t_a), \quad \hat{i}(x) = \pi_{\rho,\eta}(i_{(F_\kappa)})(x)
$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $x \in \mathcal{A}$. By the above discussions, the following relations hold:

$$
\begin{aligned}
&\sum_{\beta\in\Sigma^\rho} \widehat{S}_\beta \widehat{S}_\beta^* = 1, \qquad \hat{i}(x)\widehat{S}_\alpha \widehat{S}_\alpha^* = \widehat{S}_\alpha \widehat{S}_\alpha^* \hat{i}(x), \qquad \widehat{S}_\alpha^* \hat{i}(x) \widehat{S}_\alpha = \hat{i}(\rho_\alpha(x)), \\
&\sum_{b\in\Sigma^\eta} \widehat{T}_b \widehat{T}_b^* = 1, \qquad \hat{i}(x)\widehat{T}_a \widehat{T}_a^* = \widehat{T}_a \widehat{T}_a^* \hat{i}(x), \qquad \widehat{T}_a^* \hat{i}(x) \widehat{T}_a = \hat{i}(\eta_a(x)), \\
&\qquad \widehat{S}_\alpha \widehat{T}_b = \widehat{T}_a \widehat{S}_\beta \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta)
\end{aligned}
$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$.

For $(z, w) \in \mathbb{T}^2$, the correspondence

$$
e_{[\gamma]} \otimes P_{[\gamma]}x \in H_{(n,m)} \longrightarrow z^n w^m e_{[\gamma]} \otimes P_{[\gamma]}x \in H_{(n,m)}
$$

yields a unitary representation of $\mathbb{T}^2$ on $H_{(n,m)}$, which extends to F_κ, denoted by $u_{(z,w)}$. Since

$$
u_{(z,w)} \mathcal{T}^\kappa_{\rho,\eta} u_{(z,w)}^* = \mathcal{T}^\kappa_{\rho,\eta}, \qquad u_{(z,w)} \mathcal{I}^\kappa_{\rho,\eta} u_{(z,w)}^* = \mathcal{I}^\kappa_{\rho,\eta},
$$

The map

$$
X \in \mathcal{T}^\kappa_{\rho,\eta} \longrightarrow u_{(z,w)} X u_{(z,w)}^* \in \mathcal{T}^\kappa_{\rho,\eta}
$$

yields an action of $\mathbb{T}^2$ on the C^*-algebra $\widehat{\mathcal{O}}^\kappa_{\rho,\eta}$, which we denote by $\widehat{\theta}$. Similarly to the action θ on $\mathcal{O}^\kappa_{\rho,\eta}$, we may define the conditional expectation $\widehat{\mathcal{E}}_{\rho,\eta}$ from $\widehat{\mathcal{O}}^\kappa_{\rho,\eta}$ to the fixed point algebra $(\widehat{\mathcal{O}}^\kappa_{\rho,\eta})^{\widehat{\theta}}$ by taking the integration of the function $\widehat{\theta}_{(z,w)}(X)$ for $X \in \widehat{\mathcal{O}}^\kappa_{\rho,\eta}$. Then as in the proof of Proposition 5.8, one may prove the following theorem.

Theorem 6.7. *The algebra $\widehat{\mathcal{O}}^\kappa_{\rho,\eta}$ is canonically isomorphic to the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$ through the correspondences:*

$$
S_\alpha \longrightarrow \widehat{S}_\alpha, \qquad T_a \longrightarrow \widehat{T}_a, \qquad x \longrightarrow \hat{i}(x)
$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $x \in \mathcal{A}$.

7. K-Theory Machinery

Let us denote by $\mathcal{K}$ the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. For a C^*-algebra $\mathcal{B}$, we denote by $M(B)$ its multiplier algebra. In this section, we will study K-theory groups $K_*(\mathcal{O}^\kappa_{\rho,\eta})$ for the C^*-algebra $\mathcal{O}^\kappa_{\rho,\eta}$. We fix a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$. We define two actions

$$\hat{\rho} : \mathbb{T} \longrightarrow \mathrm{Aut}(\mathcal{O}^\kappa_{\rho,\eta}), \quad \hat{\eta} : \mathbb{T} \longrightarrow \mathrm{Aut}(\mathcal{O}^\kappa_{\rho,\eta})$$

of the circle group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ to $\mathcal{O}^\kappa_{\rho,\eta}$ by setting

$$\hat{\rho}_z = \theta_{(z,1)}, \qquad \hat{\eta}_w = \theta_{(1,w)}, \qquad z, w \in \mathbb{T}.$$

They satisfy

$$\hat{\rho}_z \circ \hat{\eta}_w = \hat{\eta}_w \circ \hat{\rho}_z = \theta_{(z,w)}, \qquad z, w \in \mathbb{T}.$$

Set the fixed point algebras

$$(\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}} = \{x \in \mathcal{O}^\kappa_{\rho,\eta} \mid \hat{\rho}_z(x) = x \text{ for all } z \in \mathbb{T}\},$$
$$(\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\eta}} = \{x \in \mathcal{O}^\kappa_{\rho,\eta} \mid \hat{\eta}_z(x) = x \text{ for all } z \in \mathbb{T}\}.$$

For $x \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}$, define the $\mathcal{O}^\kappa_{\rho,\eta}$-valued constant function $\widehat{x} \in L^1(\mathbb{T}, \mathcal{O}^\kappa_{\rho,\eta}) \subset \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T}$ from $\mathbb{T}$ by setting $\widehat{x}(z) = x, z \in \mathbb{T}$. Put $p_0 = \widehat{1}$. By [39], the algebra $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}$ is canonically isomorphic to $p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T})p_0$ through the map

$$j_\rho : x \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}} \longrightarrow \widehat{x} \in p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T})p_0$$

which induces an isomorphism

$$j_{\rho*} : K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) \longrightarrow K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T})p_0), \qquad i = 0, 1 \tag{7.1}$$

on their K-groups. By a similar manner to the proofs given in [19, Section 4], one may prove the following lemma.

Lemma 7.1.

(i) *There exists an isometry $v \in M((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T}) \otimes \mathcal{K})$ such that $vv^* = p_0 \otimes 1, v^*v = 1$.*

(ii) *$\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T}$ is stably isomorphic to $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}$, and similarly $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\eta}} \mathbb{T}$ is stably isomorphic to $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\eta}}$.*

(iii) *The inclusion $\iota_{\hat{\rho}} : p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T})p_0 \hookrightarrow \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T}$ induces an isomorphism*

$$\iota_{\hat{\rho}*} : K_0(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T})p_0) \cong K_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T})$$

on their K-groups.

Thanks to the lemma above, the isomorphism

$$Ad(v^*) : x \in p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T})p_0 \otimes \mathcal{K} \to v^*xv \in \mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T} \otimes \mathcal{K}$$

induces isomorphisms

$$Ad(v^*)_* : K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T})p_0) \longrightarrow K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T}), \qquad i = 0, 1. \tag{7.2}$$

Let $\hat{\hat{\rho}}$ be the automorphism on $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T}$ for the positive generator of $\mathbb{Z}$ for the dual action of $\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat{\rho}} \mathbb{T}$. By (7.1) and (7.2), we may define an isomorphism

$$\beta_{\rho,i} = j_{\rho*}^{-1} \circ Ad(v^*)_*^{-1} \circ \hat{\hat{\rho}}_* \circ Ad(v^*)_* \circ j_{\rho*} : K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) \longrightarrow K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}), \quad i = 0, 1$$

so that the diagram is commutative:

$$\begin{array}{ccc}
K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xrightarrow{\hat{\hat\rho}_*} & K_i(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \\
\uparrow{\scriptstyle Ad(v^*)_*} & & \uparrow{\scriptstyle Ad(v^*)_*} \\
K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) & & K_i(p_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T})p_0) \\
\uparrow{\scriptstyle j_{\rho*}} & & \uparrow{\scriptstyle j_{\rho*}} \\
K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xrightarrow{\beta_{\rho,i}} & K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}).
\end{array}$$

By [34] (cf. [12]), one has the six term exact sequence of K-theory:

$$\begin{array}{ccccc}
K_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xrightarrow{\mathrm{id}-\hat{\hat\rho}_*} & K_0(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xrightarrow{\iota_*} & K_0((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \times_{\hat{\hat\rho}} \mathbb{Z}) \\
{\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\exp} \\
K_1((\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \times_{\hat{\hat\rho}} \mathbb{Z}) & \xleftarrow[\iota_*]{} & K_1(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) & \xleftarrow[\mathrm{id}-\hat{\hat\rho}_*]{} & K_1(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}).
\end{array}$$

Since $(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \times_{\hat{\hat\rho}} \mathbb{Z} \cong \mathcal{O}^\kappa_{\rho,\eta} \otimes \mathcal{K}$ and $K_*(\mathcal{O}^\kappa_{\rho,\eta} \times_{\hat\rho} \mathbb{T}) \cong K_*((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$, one has

Lemma 7.2. *The following six term exact sequence of K-theory holds:*

$$\begin{array}{ccccc}
K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xrightarrow{\mathrm{id}-\beta_{\rho,0}} & K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xrightarrow{\iota_*} & K_0(\mathcal{O}^\kappa_{\rho,\eta}) \\
{\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\exp} \\
K_1(\mathcal{O}^\kappa_{\rho,\eta}) & \xleftarrow[\iota_*]{} & K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xleftarrow[\mathrm{id}-\beta_{\rho,1}]{} & K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}).
\end{array}$$

Hence there exist short exact sequences for $i = 0, 1$:

$$\begin{aligned}
0 &\longrightarrow \mathrm{Coker}(\mathrm{id} - \beta_{\rho,i}) \ \text{in}\ K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
&\longrightarrow K_i(\mathcal{O}^\kappa_{\rho,\eta}) \\
&\longrightarrow \mathrm{Ker}(\mathrm{id} - \beta_{\rho,i+1}) \ \text{in}\ K_{i+1}((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
&\longrightarrow 0.
\end{aligned}$$

We will then study the groups

$$\mathrm{Coker}(\mathrm{id} - \beta_{\rho,i}) \text{ in } K_i((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}), \qquad \mathrm{Ker}(\mathrm{id} - \beta_{\rho,i+1}) \text{ in } K_{i+1}((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$$

that appear in the above sequences for $i = 0, 1$. The action $\hat\eta$ acts on the subalgebra $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$, which we still denote by $\hat\eta$. Then the fixed point algebra $((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}$ of $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$ under $\hat\eta$ coincides with $\mathcal{F}_{\rho,\eta}$. The above discussions for the action $\hat\rho : \mathbb{T} \longrightarrow \mathcal{O}^\kappa_{\rho,\eta}$ works for the action $\hat\eta : \mathbb{T} \longrightarrow (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$ as in the following way. For $y \in ((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}$, define the constant function $\widehat{y} \in L^1(\mathbb{T}, (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \subset (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$ by setting $\widehat{y}(z) = y, z \in \mathbb{T}$. Putting $q_0 = \widehat{1}$, the algebra $((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}$ is canonically isomorphic to $q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0$ through the map

$$j^\rho_\eta : y \in ((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta} \longrightarrow \hat{y} \in q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0$$

which induces an isomorphism

$$j^\rho_{\eta*} : K_i(((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^{\hat\eta}) \longrightarrow K_i(q_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0)$$

on their K-groups. Similarly to Lemma 7.1, we have

Lemma 7.3.

(i) *There exists an isometry* $u \in M(((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}) \otimes \mathcal{K})$ *such that* $uu^* = q_0 \otimes 1, u^*u = 1$.

(ii) $(\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$ *is stably isomorphic to* $((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta}$.

(iii) *The inclusion* $\iota^{\hat\rho}_{\hat\eta} : q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0 (= ((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta} = \mathcal{F}_{\rho,\eta}) \hookrightarrow (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$ *induces an isomorphism*

$$\iota^{\hat\rho}_{\hat\eta *} : K_0(q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \cong K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})$$

on their K-groups.

The isomorphism

$$Ad(u^*) : y \in q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0 \longrightarrow u^*yu \in (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$$

induces isomorphisms

$$Ad(u^*)_* : K_i(q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \cong K_i((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}), \qquad i = 0, 1.$$

Let $\hat{\hat\eta}_\rho$ be the automorphism of the positive generator of $\mathbb{Z}$ for the dual action of $(\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}$. Define an isomorphism

$$\gamma_{\eta,i} = j^{\rho\,-1}_{\eta *} \circ Ad(u^*)^{-1}_* \circ \hat{\hat\eta}_{\rho *} \circ Ad(u^*)_* \circ j^{\rho}_{\eta *} : K_i(\mathcal{F}_{\rho,\eta}) \longrightarrow K_i(\mathcal{F}_{\rho,\eta})$$

such that the diagram is commutative for $i = 0, 1$:

$$\begin{tikzcd}
K_i((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}) \arrow[r, "\hat{\hat\eta}_{\rho *}"] & K_i((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T}) \\
K_i(q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \arrow[u, "Ad(u^*)_*"] & K_i(q_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho} \times_{\hat\eta} \mathbb{T})q_0) \arrow[u, "Ad(u^*)_*"'] \\
K_i(((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta}) \arrow[u, "j^{\rho}_{\eta *}"] & K_i(((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta}) \arrow[u, "j^{\rho}_{\eta *}"'] \\
K_i(\mathcal{F}_{\rho,\eta}) \arrow[u, equal] \arrow[r, "\gamma_{\eta,i}"] & K_i(\mathcal{F}_{\rho,\eta}) \arrow[u, equal]
\end{tikzcd}$$

We similarly define an endomorphism $\gamma_{\rho,i} : K_i(\mathcal{F}_{\rho,\eta}) \longrightarrow K_i(\mathcal{F}_{\rho,\eta})$ by exchanging the rôles of ρ and η.

Under the equality $((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho})^{\hat\eta} = \mathcal{F}_{\rho,\eta}$, we have the following lemma which is similar to Lemma 6.2

Lemma 7.4. *The following six term exact sequence of K-theory holds:*

$$\begin{tikzcd}
K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\mathrm{id}-\gamma_{\eta,0}"] & K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\iota_*"] & K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \arrow[d, "\exp"] \\
K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \arrow[u, "\delta"] & K_1(\mathcal{F}_{\rho,\eta}) \arrow[l, "\iota_*"] & K_1(\mathcal{F}_{\rho,\eta}). \arrow[l, "\mathrm{id}-\gamma_{\eta,1}"]
\end{tikzcd}$$

In particular, if $K_1(\mathcal{F}_{\rho,\eta}) = 0$, *we have*

$$K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) = \mathrm{Coker}(\mathrm{id} - \gamma_{\eta,0}) \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}),$$
$$K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) = \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \quad \textit{in } K_0(\mathcal{F}_{\rho,\eta}).$$

Denote by $M_n(\mathcal{B})$ the $n\times n$ matrix algebra over a C^*-algebra $\mathcal{B}$, which is identified with the tensor product $\mathcal{B}\otimes M_n(\mathcal{C})$. The following lemmas hold.

Lemma 7.5. *For a projection $q \in M_n((\mathcal{O}^\kappa_{\rho,\eta})^\rho)$ and a partial isometry $S \in \mathcal{O}^\kappa_{\rho,\eta}$ such that*

$$\hat\rho_z(S) = zS \quad \text{for } z \in \mathbb{T}, \qquad q(SS^*\otimes 1_n) = (SS^*\otimes 1_n)q,$$

we have

$$\beta^{-1}_{\rho,0}([(SS^*\otimes 1_n)q]) = [(S^*\otimes 1_n)q(S\otimes 1_n)] \quad \text{in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}).$$

Proof. As q commutes with $SS^*\otimes 1_n$, $p = (S^*\otimes 1_n)q(S\otimes 1_n)$ is a projection in $(\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$. Since $p \le S^*S\otimes 1_n$, By a similar argument to the proof of [19, Lemma 4.5], one sees that $\beta_{\rho,0}([p]) = [(S\otimes 1_n)p(S^*\otimes 1_n)]$ in $K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$. □

Lemma 7.6.

(i) *For a projection $q \in M_n(\mathcal{F}_{\rho,\eta})$ and a partial isometry $T \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$ such that*

$$\hat\eta_z(T) = zT \quad \text{for } z \in \mathbb{T}, \qquad q(TT^*\otimes 1_n) = (TT^*\otimes 1_n)q,$$

we have

$$\gamma^{-1}_{\eta,0}([(TT^*\otimes 1_n)q]) = [(T^*\otimes 1_n)q(T\otimes 1_n)] \quad \text{in } K_0(\mathcal{F}_{\rho,\eta}).$$

(ii) *For a projection $q \in M_n(\mathcal{F}_{\rho,\eta})$ and a partial isometry $S \in (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\eta}$ such that*

$$\hat\rho_z(S) = zS \quad \text{for } z \in \mathbb{T}, \qquad q(SS^*\otimes 1_n) = (SS^*\otimes 1_n)q,$$

we have

$$\gamma^{-1}_{\rho,0}([(SS^*\otimes 1_n)q]) = [(S^*\otimes 1_n)q(S\otimes 1_n)] \quad \text{in } K_0(\mathcal{F}_{\rho,\eta}).$$

Hence we have

Lemma 7.7. *The diagram*

$$\begin{array}{ccc}
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\mathrm{id}-\gamma_{\rho,0}} & K_0(\mathcal{F}_{\rho,\eta}) \\
\Big\downarrow \iota_* & & \Big\downarrow \iota_* \\
K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) & \xrightarrow{\mathrm{id}-\beta_{\rho,0}} & K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})
\end{array}$$

is commutative.

Proof. By [29, Proposition 3.3], the map $\iota_* : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$ is induced by the natural inclusion $\mathcal{F}_{\rho,\eta}(= ((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})^\eta) \hookrightarrow (\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}$. For an element $[q] \in K_0(\mathcal{F}_{\rho,\eta})$ one may assume that $q \in M_n(\mathcal{F}_{\rho,\eta})$ for some $n \in \mathbb{N}$ so that one has

$$\begin{aligned}
\gamma^{-1}_{\rho,0}([q]) &= \sum_{\alpha\in\Sigma^\rho} [(S_\alpha S^*_\alpha \otimes 1_n)q] \\
&= \sum_{\alpha\in\Sigma^\rho} [(S^*_\alpha \otimes 1_n)q(S_\alpha\otimes 1_n)] \\
&= \sum_{\alpha\in\Sigma^\rho} \beta^{-1}_{\rho,0}([q(S_\alpha S^*_\alpha\otimes 1_n)]) = \beta^{-1}_{\rho,0}([q])
\end{aligned}$$

so that $\beta_{\rho,0}|_{K_0(\mathcal{F}_{\rho,\eta})} = \gamma_{\rho,0}$. □

In the rest of this section, we assume that $K_1(\mathcal{F}_{\rho,\eta}) = 0$. The following lemma is crucial in our further discussions.

Lemma 7.8. *In the six term exact sequence in Lemma 7.4 with $K_1(\mathcal{F}_{\rho,\eta}) = 0$, we have the following commutative diagrams:*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \arrow[r, "\mathrm{id}-\beta_{\rho,1}"] \arrow[d, "\delta"'] & K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \arrow[d, "\delta"'] \\
K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\mathrm{id}-\gamma_{\rho,0}"] \arrow[d, "\mathrm{id}-\gamma_{\eta,0}"'] & K_0(\mathcal{F}_{\rho,\eta}) \arrow[d, "\mathrm{id}-\gamma_{\eta,0}"'] \\
K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\mathrm{id}-\gamma_{\rho,0}"] \arrow[d, "\iota_*"'] & K_0(\mathcal{F}_{\rho,\eta}) \arrow[d, "\iota_*"'] \\
K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \arrow[r, "\mathrm{id}-\beta_{\rho,0}"] \arrow[d] & K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \arrow[d] \\
0 & 0
\end{tikzcd} \tag{7.3}$$

Proof. It is well-known that δ-map is functorial (see [42, Theorem 7.2.5], [3, p.266 (LX)]). Hence the diagram of the upper square

$$\begin{tikzcd}
K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \arrow[r, "\mathrm{id}-\beta_{\rho,1}"] \arrow[d, "\delta"'] & K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \arrow[d, "\delta"'] \\
K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\mathrm{id}-\gamma_{\rho,0}"] & K_0(\mathcal{F}_{\rho,\eta})
\end{tikzcd}$$

is commutative. Since $\gamma_{\rho,0} \circ \gamma_{\eta,0} = \gamma_{\eta,0} \circ \gamma_{\rho,0}$ the diagram of the middle square

$$\begin{tikzcd}
K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\mathrm{id}-\gamma_{\rho,0}"] \arrow[d, "\mathrm{id}-\gamma_{\eta,0}"] & K_0(\mathcal{F}_{\rho,\eta}) \arrow[d, "\mathrm{id}-\gamma_{\eta,0}"] \\
K_0(\mathcal{F}_{\rho,\eta}) \arrow[r, "\mathrm{id}-\gamma_{\rho,0}"] & K_0(\mathcal{F}_{\rho,\eta})
\end{tikzcd} \tag{7.4}$$

is commutative. The commutativity of the lower square comes from the preceding lemma. ☐

We will describe the K-groups $K_*(\mathcal{O}^{\kappa}_{\rho,\eta})$ in terms of the kernels and cokernels of the homomorphisms $\mathrm{id} - \gamma_{\rho,0}$ and $\mathrm{id} - \gamma_{\eta,0}$ on $K_0(\mathcal{F}_{\rho,\eta})$. Recall that there exist two short exact sequences by Lemma 7.2:

$$\begin{aligned}
0 &\longrightarrow \mathrm{Coker}(\mathrm{id} - \beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \\
&\longrightarrow K_0(\mathcal{O}^{\kappa}_{\rho,\eta}) \\
&\longrightarrow \mathrm{Ker}(\mathrm{id} - \beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat\rho}) \\
&\longrightarrow 0
\end{aligned}$$

and

$$\begin{aligned}
0 &\longrightarrow \mathrm{Coker}(\mathrm{id}-\beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
&\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta}) \\
&\longrightarrow \mathrm{Ker}(\mathrm{id}-\beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
&\longrightarrow 0.
\end{aligned}$$

As $\gamma_{\eta,0}\circ\gamma_{\rho,0} = \gamma_{\rho,0}\circ\gamma_{\eta,0}$ on $K_0(\mathcal{F}_{\rho,\eta})$, the homomorphisms $\gamma_{\rho,0}$ and $\gamma_{\eta,0}$ naturally act on $\mathrm{Coker}(\mathrm{id}-\gamma_{\eta,0}) = K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$ and $\mathrm{Coker}(\mathrm{id}-\gamma_{\rho,0}) = K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})$ as endomorphisms respectively, which we denote by $\bar\gamma_{\rho,0}$ and $\bar\gamma_{\eta,0}$ respectively.

Lemma 7.9.

(i) *For $K_0(\mathcal{O}^\kappa_{\rho,\eta})$, we have*

$$\begin{aligned}
&\mathrm{Coker}(\mathrm{id}-\beta_{\rho,0}) \textit{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
\cong&\mathrm{Coker}(\mathrm{id}-\bar\gamma_{\rho,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) \\
\cong&K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}))
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Ker}(\mathrm{id}-\beta_{\rho,1}) \textit{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
\cong&\mathrm{Ker}(\mathrm{id}-\gamma_{\rho,0}) \textit{ in } (\mathrm{Ker}(\mathrm{id}-\gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta})) \\
\cong&\mathrm{Ker}(\mathrm{id}-\gamma_{\rho,0}) \cap \mathrm{Ker}(\mathrm{id}-\gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta}).
\end{aligned}$$

(ii) *For $K_1(\mathcal{O}^\kappa_{\rho,\eta})$, we have*

$$\begin{aligned}
&\mathrm{Coker}(\mathrm{id}-\beta_{\rho,1}) \textit{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
\cong&(\mathrm{Ker}(\mathrm{id}-\gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id}-\gamma_{\rho,0})(\mathrm{Ker}(\mathrm{id}-\gamma_{\eta,0}) \textit{ in } K_0(\mathcal{F}_{\rho,\eta}))
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Ker}(\mathrm{id}-\beta_{\rho,0}) \textit{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
\cong&\mathrm{Ker}(\mathrm{id}-\bar\gamma_{\rho,0}) \textit{ in } (K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})).
\end{aligned}$$

Proof. (i) We will first prove the assertions for the group $\mathrm{Coker}(\mathrm{id}-\beta_{\rho,0})$ in $K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho})$. In the diagram (7.3), the exactness of the vertical arrows implies that ι_* is surjective so that

$$K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \cong \iota_*(K_0(\mathcal{F}_{\rho,\eta})) \cong K_0(\mathcal{F}_{\rho,\eta})/\mathrm{Ker}(\mathrm{id}-\gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}).$$

By the commutativity in the lower square in the diagram (7.3), one has

$$\begin{aligned}
&\mathrm{Coker}(\mathrm{id}-\beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat\rho}) \\
\cong&\mathrm{Coker}(\mathrm{id}-\bar\gamma_{\rho,0}) \text{ in } (\mathrm{Coker}(\mathrm{id}-\gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}).)
\end{aligned}$$

The latter group will be proved to be isomorphic to the group

$$K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) + (\mathrm{id}-\gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})).$$

Put $H_{\rho,\eta} = (\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})$ the subgroup of $K_0(\mathcal{F}_{\rho,\eta})$ generated by $(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$ and $(\mathrm{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})$. Set the quotient maps

$$
\begin{aligned}
K_0(\mathcal{F}_{\rho,\eta}) &\xrightarrow{q_\eta} K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) \\
&\xrightarrow{q_{(\mathrm{id}-\gamma_{\rho,0})}} \mathrm{Coker}(\mathrm{id} - \bar{\gamma}_{\rho,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})
\end{aligned}
$$

and $\Phi = q_{(\mathrm{id}-\gamma_{\rho,0})} \circ q_\eta : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow \mathrm{Coker}(\mathrm{id}-\bar{\gamma}_{\rho,0})$ in $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$. It suffices to show the equality $\mathrm{Ker}(\Phi) = H_{\rho,\eta}$. As $(\mathrm{id} - \gamma_{\rho,0})$ commutes with $(\mathrm{id} - \gamma_{\eta,0})$, one has

$$
(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}) \subset \mathrm{Ker}(\Phi), \qquad (\mathrm{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) \subset \mathrm{Ker}(\Phi).
$$

Hence we have $H_{\rho,\eta} \subset \mathrm{Ker}(\Phi)$. On the other hand, for $g \in \mathrm{Ker}(\Phi)$, we have $g \in (\mathrm{id}-\bar{\gamma}_{\rho,0})(K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta}))$ so that $g = (\mathrm{id}-\gamma_{\rho,0})[h]$ for some $[h] \in K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$. Hence $g = (\mathrm{id}-\gamma_{\rho,0})h + (\mathrm{id}-\gamma_{\rho,0})(\mathrm{id}-\gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})$ so that $g \in H_{\rho,\eta}$. Hence we have $\mathrm{Ker}(\Phi) \subset H_{\rho,\eta}$ and $\mathrm{Ker}(\Phi) = H_{\rho,\eta}$.

We will second prove the assertions for the group $\mathrm{Ker}(\mathrm{id} - \beta_{\rho,1})$ in $K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}})$. In the diagram (7.3), the exactness of the vertical arrows implies that δ is injective and $\mathrm{Im}(\delta) = \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0})$ so that we have

$$
K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) \cong \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}). \tag{7.5}
$$

By the commutativity in the upper square in the diagram (7.3), one has

$$
\mathrm{Ker}(\mathrm{id} - \beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\rho}}) \cong \mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \text{ in } (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})).
$$

Since $\gamma_{\eta,0}$ commutes with $\gamma_{\rho,0}$ in $K_0(\mathcal{F}_{\rho,\eta})$, we have

$$
\begin{aligned}
&\mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \text{ in } (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})) \\
\cong &\mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \cap \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}).
\end{aligned}
$$

(ii) The assertions are similarly shown to (i). ☐

Therefore we have

Theorem 7.10. *Assume that $K_1(\mathcal{F}_{\rho,\eta}) = 0$. There exist short exact sequences:*

$$
\begin{aligned}
0 &\longrightarrow K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) \\
&\longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta}) \\
&\longrightarrow \mathrm{Ker}(\mathrm{id} - \gamma_{\rho,0}) \cap \mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}) \\
&\longrightarrow 0
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\longrightarrow (\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id} - \gamma_{\rho,0})(\mathrm{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})) \\
&\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta}) \\
&\longrightarrow \mathrm{Ker}(\mathrm{id} - \bar{\gamma}_{\rho,0}) \text{ in } (K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id} - \gamma_{\eta,0})K_0(\mathcal{F}_{\rho,\eta})) \\
&\longrightarrow 0.
\end{aligned}
$$

We may describe the above formulae as follows.

Corollary 7.11. *Suppose $K_1(\mathcal{F}_{\rho,\eta}) = 0$. There exist short exact sequences:*

$$
\begin{aligned}
0 &\longrightarrow \operatorname{Coker}(\mathrm{id} - \bar{\gamma}_{\rho,0}) \text{ in } (\operatorname{Coker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))\\
&\longrightarrow K_0(\mathcal{O}^{\kappa}_{\rho,\eta})\\
&\longrightarrow \operatorname{Ker}(\mathrm{id} - \gamma_{\rho,0}) \text{ in } (\operatorname{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))\\
&\longrightarrow 0
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\longrightarrow \operatorname{Coker}(\mathrm{id} - \gamma_{\rho,0}) \text{ in } ((\operatorname{Ker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))\\
&\longrightarrow K_1(\mathcal{O}^{\kappa}_{\rho,\eta})\\
&\longrightarrow \operatorname{Ker}(\mathrm{id} - \bar{\gamma}_{\rho,0}) \text{ in } (\operatorname{Coker}(\mathrm{id} - \gamma_{\eta,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))\\
&\longrightarrow 0.
\end{aligned}
$$

8. K-Theory formulae

We henceforth denote the endomorphisms $\gamma_{\rho,0}, \gamma_{\eta,0}$ on $K_0(\mathcal{F}_{\rho,\eta})$ by γ_ρ, γ_η respectively. In this section, we will present more useful formulae to compute the K-groups $K_i(\mathcal{O}^{\kappa}_{\rho,\eta})$ under a certain additional assumption on $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$. The assumed condition on $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is the following:
Definition. A C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to *form square* if the C^*-subalgebra $C^*(\rho_\alpha(1) : \alpha \in \Sigma^\rho)$ of $\mathcal{A}$ generated by the projections $\rho_\alpha(1), \alpha \in \Sigma^\rho$ coincides with the C^*-subalgebra $C^*(\eta_a(1) : a \in \Sigma^\eta)$ of $\mathcal{A}$ generated by the projections $\eta_a(1), a \in \Sigma^\eta$.

Lemma 8.1. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square. Put for $l \in \mathbb{Z}_+$*

$$\mathcal{A}^\rho_l = C^*(\rho_\mu(1) : \mu \in B_l(\Lambda_\rho)), \qquad \mathcal{A}^\eta_l = C^*(\eta_\xi(1) : \xi \in B_l(\Lambda_\eta)).$$

Then $\mathcal{A}^\rho_l = \mathcal{A}^\eta_l$.

Proof. By the assumption, we have $\mathcal{A}^\rho_1 = \mathcal{A}^\eta_1$. Hence the desired equality for $l = 1$ holds. Suppose that the equalities hold for all $l \le k$ for some $k \in \mathbb{N}$. For $\mu = \mu_1\mu_2\cdots\mu_k\mu_{k+1} \in B_{k+1}(\Lambda_\rho)$ we have $\rho_\mu(1) = \rho_{\mu_{k+1}}(\rho_{\mu_1\mu_2\cdots\mu_k}(1))$ so that $\rho_\mu(1) \in \rho_{\mu_{k+1}}(\mathcal{A}^\rho_k)$. By the commutation relation (3.1), one sees that

$$\rho_{\mu_{k+1}}(\mathcal{A}^\rho_k) \subset C^*(\eta_\xi(\rho_\alpha(1)) : \xi \in B_k(\Lambda_\eta), \alpha \in \Sigma^\rho).$$

Since $C^*(\rho_\alpha(1) : \alpha \in \Sigma^\rho) = C^*(\eta_a(1) : a \in \Sigma^\eta)$, the algebra $C^*(\eta_\xi(\rho_\alpha(1)) : \xi \in B_k(\Lambda_\eta), \alpha \in \Sigma^\rho)$ is contained in $\mathcal{A}^\eta_{k+1}$ so that $\rho_{\mu_{k+1}}(\mathcal{A}^\eta_k) \subset \mathcal{A}^\eta_{k+1}$. This implies $\rho_\mu(1) \in \mathcal{A}^\eta_{k+1}$ so that $\mathcal{A}^\rho_{k+1} \subset \mathcal{A}^\eta_{k+1}$ and hence $\mathcal{A}^\rho_{k+1} = \mathcal{A}^\eta_{k+1}$. □

Therefore we have

Lemma 8.2. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square. Put for $j, k \in \mathbb{Z}_+$*

$$
\begin{aligned}
\mathcal{A}_{j,k} &= C^*(\rho_\mu(\eta_\zeta(1)) : \mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta))\\
&\quad (= C^*(\eta_\xi(\rho_\nu(1)) : \xi \in B_k(\Lambda_\eta), \nu \in B_j(\Lambda_\rho))).
\end{aligned}
$$

Then $\mathcal{A}_{j,k}$ is commutative and of finite dimensional such that

$$\mathcal{A}_{j,k} = \mathcal{A}^\rho_{j+k} (= \mathcal{A}^\eta_{j+k}).$$

Hence $\mathcal{A}_{j,k} = \mathcal{A}_{j',k'}$ if $j + k = j' + k'$.

Proof. Since $\eta_\zeta(1) \in Z_{\mathcal{A}}$ and $\rho_\mu(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$, the algebra $\mathcal{A}_{j,k}$ belongs to the center $Z_{\mathcal{A}}$ of $\mathcal{A}$. By the preceding lemma, we have

$$\mathcal{A}_{j,k} = C^*(\rho_\mu(\rho_\nu(1)) : \mu \in B_j(\Lambda_\rho), \nu \in B_k(\Lambda_\rho)) = \mathcal{A}^\rho_{j+k}.$$

□

For $j,k \in \mathbb{Z}_+$, put $l = j+k$. We denote by $\mathcal{A}_l$ the commutative finite dimensional algebra $\mathcal{A}_{j,k}$. Put $m(l) = \dim \mathcal{A}_l$. Take the finite sequence of minimal projections $E_i^l, i = 1,2,\dots,m(l)$ in $\mathcal{A}_l$ such that $\sum_{i=1}^{m(l)} E_i^l = 1$. Hence we have $\mathcal{A}_l = \oplus_{i=1}^{m(l)} \mathbb{C}E_i^l$. Since $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$, there exists $A^\rho_{l,l+1}(i,\alpha,n)$, which takes 0 or 1, such that

$$\rho_\alpha(E_i^l) = \sum_{n=1}^{m(l+1)} A^\rho_{l,l+1}(i,\alpha,n)E_n^{l+1}, \qquad \alpha \in \Sigma^\rho,\ i = 1,\dots,m(l).$$

Similarly, there exists $A^\eta_{l,l+1}(i,a,n)$, which takes 0 or 1, such that

$$\eta_a(E_i^l) = \sum_{n=1}^{m(l+1)} A^\eta_{l,l+1}(i,a,n)E_n^{l+1}, \qquad a \in \Sigma^\eta,\ i = 1,\dots,m(l).$$

Set for $i = 1,\dots,m(l)$

$$\begin{aligned}\mathcal{F}_{j,k}(i) &= C^*(S_\mu T_\zeta E_i^l x E_i^l T_\xi^* S_\nu^* \mid \mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta), x \in \mathcal{A}),\\ &= C^*(T_\zeta S_\mu E_i^l x E_i^l S_\nu^* T_\xi^* \mid \mu,\nu \in B_j(\Lambda_\rho), \zeta,\xi \in B_k(\Lambda_\eta), x \in \mathcal{A}).\end{aligned}$$

Let $N_{j,k}(i)$ be the cardinal number of the finite set

$$\{(\mu,\zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta) \mid \rho_\mu(\eta_\zeta(1)) \ge E_i^l\}.$$

Since E_i^l is a central projection in $\mathcal{A}$, we have

Lemma 8.3. *For $j,k \in \mathbb{Z}_+$, put $l = j+k$. Then we have*

(i) *$\mathcal{F}_{j,k}(i)$ is isomorphic to the matrix algebra $M_{N_{j,k}(i)}(E_i^l\mathcal{A}E_i^l)(= M_{N_{j,k}(i)}(\mathbb{C}) \otimes E_i^l\mathcal{A}E_i^l)$ over $E_i^l\mathcal{A}E_i^l$ for $i = 1,\dots,m(l)$.*

(ii) *$\mathcal{F}_{j,k} = \mathcal{F}_{j,k}(1) \oplus \cdots \oplus \mathcal{F}_{j,k}(m(l))$.*

Proof. (i) For $(\mu,\zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta)$ with $S_\mu T_\zeta E_i^l \ne 0$, one has $\eta_\zeta(\rho_\mu(1))E_i^l \ne 0$ so that $\eta_\zeta(\rho_\mu(1)) \ge E_i^l$. Hence $(S_\mu T_\zeta E_i^l)^* S_\mu T_\zeta E_i^l = E_i$. One sees that the set

$$\{S_\mu T_\zeta E_i^l \mid (\mu,\zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta); S_\mu T_\zeta E_i^l \ne 0\}$$

consist of partial isometries which give rise to matrix units of $\mathcal{F}_{j,k}(i)$ such that $\mathcal{F}_{j,k}(i)$ is isomorphic to $M_{N_{j,k}(i)}(E_i^l\mathcal{A}E_i^l)$.

(ii) Since $\mathcal{A} = E_1^l\mathcal{A}E_1^l \oplus \cdots \oplus E_{m(l)}^l\mathcal{A}E_{m(l)}^l$, the assertion is easy. □

Define $\lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ by setting

$$\lambda_\rho([p]) = \sum_{\alpha \in \Sigma^\rho} [\rho_\alpha \otimes 1_n(p)], \qquad \lambda_\eta([p]) = \sum_{a \in \Sigma^\eta} [\eta_a \otimes 1_n(p)]$$

for a projection $p \in M_n(\mathcal{A})$ for some $n \in \mathbb{N}$. Recall that the identities (5.1), (5.2) give rise to the embeddings (5.3), which induces homomorphisms

$$K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j,k+1}), \qquad K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j+1,k}).$$

We still denote them by $\iota_{*,+1}, \iota_{+1,*}$ respectively.

Lemma 8.4. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square. There exists an isomorphism*

$$\Phi_{j,k} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{A})$$

such that the following diagrams are commutative:

(i)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) \\ {\scriptstyle \Phi_{j,k}}\downarrow & & {\scriptstyle \Phi_{j+1,k}}\downarrow \\ K_0(\mathcal{A}) & \xrightarrow{\lambda_\rho} & K_0(\mathcal{A}) \end{array}$$

(ii)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{j,k+1}) \\ {\scriptstyle \Phi_{j,k}}\downarrow & & {\scriptstyle \Phi_{j,k+1}}\downarrow \\ K_0(\mathcal{A}) & \xrightarrow{\lambda_\eta} & K_0(\mathcal{A}) \end{array}$$

Proof. Put for $i = 1, 2, \cdots, m(l)$

$$P_i = \sum_{\mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta)} S_\mu T_\zeta E_i^l T_\zeta^* S_\mu^*.$$

Then P_i is a central projection in $\mathcal{F}_{j,k}$ such that $\sum_{i=1}^{m(l)} P_i = 1$. For $X \in \mathcal{F}_{j,k}$, one has $P_i X P_i \in \mathcal{F}_{j,k}(i)$ such that

$$X = \sum_{i=1}^{m(l)} P_i X P_i \in \bigoplus_{i=1}^{m(l)} \mathcal{F}_{j,k}(i).$$

Define an isomorphism

$$\varphi_{j,k} : X \in \mathcal{F}_{j,k} \longrightarrow \sum_{i=1}^{m(l)} P_i X P_i \in \bigoplus_{i=1}^{m(l)} \mathcal{F}_{j,k}(i)$$

which induces an isomorphism on their K-groups

$$\varphi_{j,k*} : K_0(\mathcal{F}_{j,k}) \longrightarrow \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)).$$

Take and fix $\nu(i), \mu(i) \in B_j(\Lambda_\rho)$ and $\zeta(i), \xi(i) \in B_k(\Lambda_\eta)$ such that

$$T_{\xi(i)} S_{\nu(i)} = S_{\mu(i)} T_{\zeta(i)} \quad \text{and} \quad T_{\xi(i)} S_{\nu(i)} E_i^l \neq 0. \tag{8.1}$$

Hence $S_{\nu(i)}^* T_{\xi(i)}^* T_{\xi(i)} S_{\nu(i)} \geq E_i^l$. Since $\mathcal{F}_{j,k}(i)$ is isomorphic to $M_{N_{j,k(i)}}(\mathbb{C}) \otimes E_i^l \mathcal{A} E_i^l$, the embedding

$$\iota_{j,k}(i) : x \in E_i^l \mathcal{A} E_i^l \longrightarrow T_{\xi(i)} S_{\nu(i)} x S_{\nu(i)}^* T_{\xi(i)}^* \in \mathcal{F}_{j,k}(i)$$

induces an isomorphism on their K-groups

$$\iota_{j,k}(i)_* : K_0(E_i^l \mathcal{A} E_i^l) \longrightarrow K_0(\mathcal{F}_{j,k}(i)).$$

Put

$$\psi_{j,k} = \bigoplus_{i=1}^{m(l)} \iota_{j,k}(i) : \bigoplus_{i=1}^{m(l)} E_i^l \mathcal{A} E_i^l \longrightarrow \bigoplus_{i=1}^{m(l)} \mathcal{F}_{j,k}(i)$$

and hence we have an isomorphism

$$\psi_{j,k*} = \bigoplus_{i=1}^{m(l)} \iota_{j,k}(i)_* : \bigoplus_{i=1}^{m(l)} K_0(E_i^l \mathcal{A} E_i^l) \longrightarrow \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)).$$

We then have isomorphisms

$$K_0(\mathcal{F}_{j,k}) \xrightarrow{\varphi_{j,k*}} \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)) \xrightarrow{{\psi_{j,k*}}^{-1}} \bigoplus_{i=1}^{m(l)} K_0(E_i^l \mathcal{A} E_i^l).$$

Since $K_0(\mathcal{A}) = \bigoplus_{i=1}^{m(l)} K_0(E_i^l \mathcal{A} E_i^l)$, we have an isomorphism

$$\Phi_{j,k} = {\psi_{j,k*}}^{-1} \circ \varphi_{j,k*} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{A}).$$

(i) It suffices to show the following diagram

$$\begin{array}{ccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) \\
{\scriptstyle \varphi_{j,k*}} \big\downarrow & & {\scriptstyle \varphi_{j+1,k*}} \big\downarrow \\
\bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)) & & \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j+1,k}(i)) \\
{\scriptstyle \psi_{j,k*}} \big\uparrow & & {\scriptstyle \psi_{j+1,k*}} \big\uparrow \\
K_0(\mathcal{A}) & \xrightarrow{\lambda_\rho} & K_0(\mathcal{A})
\end{array}$$

is commutative. For $x = \sum_{i=1}^{m(l)} E_i^l x E_i^l \in \mathcal{A}$, we have

$$\psi_{j,k}(x) = \sum_{i=1}^{m(l)} T_{\xi(i)} S_{\nu(i)} E_i^l x E_i^l S_{\nu(i)}^* T_{\xi(i)}^* = \sum_{i=1}^{m(l)} S_{\mu(i)} T_{\zeta(i)} E_i^l x E_i^l T_{\zeta(i)}^* S_{\mu(i)}^*.$$

Since $P_i T_{\xi(i)} S_{\nu(i)} E_i^l x E_i^l S_{\nu(i)}^* T_{\xi(i)}^* P_i = T_{\xi(i)} S_{\nu(i)} E_i^l x E_i^l S_{\nu(i)}^* T_{\xi(i)}^*$, we have

$$\varphi_{j,k}^{-1} \circ \psi_{j,k}(x) = \sum_{i=1}^{m(l)} T_{\xi(i)} S_{\nu(i)} E_i^l x E_i^l S_{\nu(i)}^* T_{\xi(i)}^*$$

so that

$$\iota_{+1,*} \circ \varphi_{j,k}^{-1} \circ \psi_{j,k}(x) = \sum_{\alpha \in \Sigma^\rho} \sum_{i=1}^{m(l)} T_{\xi(i)} S_{\nu(i)\alpha} \rho_\alpha(E_i^l x E_i^l) S_{\nu(i)\alpha}^* T_{\xi(i)}^*.$$

Since

$$S_{\nu(i)\alpha} \rho_\alpha(E_i^l x E_i^l) S_{\nu(i)\alpha}^* = \sum_{n=1}^{m(l+1)} A_{l,l+1}^\rho(i,\alpha,n) S_{\nu(i)\alpha} E_n^{l+1} \rho_\alpha(x) E_n^{l+1} S_{\nu(i)\alpha}^*$$

and $A_{l,l+1}^\rho(i,\alpha,n) S_{\nu(i)\alpha} E_n^{l+1} = S_{\nu(i)\alpha} E_n^{l+1}$, we have

$$\sum_{\alpha \in \Sigma^\rho} S_{\nu(i)\alpha} \rho_\alpha(E_i^l x E_i^l) S_{\nu(i)\alpha}^* = \sum_{n=1}^{m(l+1)} \sum_{\alpha \in \Sigma^\rho} S_{\nu(i)\alpha} E_n^{l+1} \rho_\alpha(x) E_n^{l+1} S_{\nu(i)\alpha}^*$$

so that

$$\iota_{+1,*} \circ \varphi_{j,k}^{-1} \circ \psi_{j,k}(x) = \sum_{\alpha \in \Sigma^\rho} \sum_{i=1}^{m(l)} \sum_{n=1}^{m(l+1)} T_{\xi(i)} S_{\nu(i)\alpha} E_n^{l+1} \rho_\alpha(x) E_n^{l+1} S_{\nu(i)\alpha}^* T_{\xi(i)}^*.$$

On the other hand,

$$
\begin{aligned}
\psi_{j,k}(\lambda_\rho(x)) &= \psi_{j,k}(\sum_{\alpha\in\Sigma^\rho}\rho_\alpha(x)) \\
&= \psi_{j,k}(\sum_{n=1}^{m(l+1)}\sum_{\alpha\in\Sigma^\rho}E_n^{l+1}\rho_\alpha(x))E_n^{l+1} \\
&= \sum_{\alpha\in\Sigma^\rho}\sum_{i=1}^{m(l)}\sum_{n=1}^{m(l+1)}T_{\xi(i)}S_{\nu(i)\alpha}E_n^{l+1}\rho_\alpha(x)E_n^{l+1}S^*_{\nu(i)\alpha}T^*_{\xi(i)}.
\end{aligned}
$$

Therefore we have

$$\iota_{+1,*}\circ\varphi_{j,k}^{-1}\circ\psi_{j,k}(x) = \psi_{j,k}(\lambda_\rho(x)).$$

(ii) is symmetric to (i). □

Define the abelian groups of the inductive limits:

$$G_\rho = \lim\{\lambda_\rho : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}, \qquad G_\eta = \lim\{\lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}.$$

Put the subalgebras of $\mathcal{F}_{\rho,\eta}$ for $j,k\in\mathbb{Z}_+$

$$
\begin{aligned}
\mathcal{F}_{\rho,k} &= C^*(T_\zeta S_\mu x S_\nu^* T_\xi^* \mid \mu,\nu\in B_*(\Lambda_\rho), |\mu|=|\nu|, \zeta,\xi\in B_k(\Lambda_\eta), x\in\mathcal{A}) \\
&= C^*(T_\zeta y T_\xi^* \mid \zeta,\xi\in B_k(\Lambda_\eta), y\in\mathcal{F}_\rho), \\
\mathcal{F}_{j,\eta} &= C^*(S_\mu T_\zeta x T_\xi^* S_\nu^* \mid \mu,\nu\in B_j(\Lambda_\rho), \zeta,\xi\in B_*(\Lambda_\eta), |\zeta|=|\xi|, x\in\mathcal{A}) \\
&= C^*(S_\mu y S_\nu^* \mid \mu,\nu\in B_j(\Lambda_\rho), y\in\mathcal{F}_\eta).
\end{aligned}
$$

By the preceding lemma, we have

Lemma 8.5. *For $j,k\in\mathbb{Z}_+$, there exist isomorphisms*

$$\Phi_{\rho,k} : K_0(\mathcal{F}_{\rho,k}) \longrightarrow G_\rho, \qquad \Phi_{j,\eta} : K_0(\mathcal{F}_{j,\eta}) \longrightarrow G_\eta$$

such that the following diagrams are commutative:

(i)

$$
\begin{array}{ccccccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) & \xrightarrow{\iota_{+1,*}} & \cdots & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{\rho,k}) \\
{\scriptstyle\Phi_{j,k}}\downarrow & & {\scriptstyle\Phi_{j+1,k}}\downarrow & & & & {\scriptstyle\Phi_{\rho,k}}\downarrow \\
K_0(\mathcal{A}) & \xrightarrow{\lambda_\rho} & K_0(\mathcal{A}) & \xrightarrow{\lambda_\rho} & \cdots & \xrightarrow{\lambda_\rho} & G_\rho
\end{array}
$$

(ii)

$$
\begin{array}{ccccccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{j,k+1}) & \xrightarrow{\iota_{*,+1}} & \cdots & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{j,\eta}) \\
{\scriptstyle\Phi_{j,k}}\downarrow & & {\scriptstyle\Phi_{j,k+1}}\downarrow & & & & {\scriptstyle\Phi_{j,\eta}}\downarrow \\
K_0(\mathcal{A}) & \xrightarrow{\lambda_\eta} & K_0(\mathcal{A}) & \xrightarrow{\lambda_\eta} & \cdots & \xrightarrow{\lambda_\eta} & G_\eta.
\end{array}
$$

Lemma 8.6. *If $\xi=\xi_1\cdots\xi_k\in B_k(\Lambda_\eta), \nu=\nu_1\cdots\nu_j\in B_j(\Lambda_\rho)$ and $i=1,\dots,m(l)$ satisfy the condition $\rho_\nu(\eta_\xi(1))\geq E_i^l$ where $l=j+k$, then $T^*_{\xi_1}T_\xi S_\nu E_i^l = T_{\bar{\xi}}S_\nu E_i^l$ where $\bar{\xi}=\xi_2\cdots\xi_k$.*

Proof. Since $T^*_{\xi_1}T_\xi = T^*_{\xi_1}T_{\xi_1}T_{\bar{\xi}}T^*_{\bar{\xi}}T_{\bar{\xi}} = T_{\bar{\xi}}T^*_{\bar{\xi}}T^*_{\xi_1}T_{\xi_1}T_{\bar{\xi}} = T_{\bar{\xi}}T^*_{\xi}T_\xi$, we have

$$T^*_{\xi_1}T_\xi S_\nu E_i^l = T_{\bar{\xi}}S_\nu S_\nu^* T_\xi^* T_\xi S_\nu E_i^l = T_{\bar{\xi}}S_\nu\rho_\nu(\eta_\xi(1))E_i^l = T_{\bar{\xi}}S_\nu E_i^l.$$

□

Lemma 8.7. *For $k, j \in \mathbb{Z}_+$, we have*

(i) *The restriction of γ_η^{-1} to $K_0(\mathcal{F}_{j,k})$ makes the following diagram commutative:*

$$\begin{array}{ccccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{j,k-1}) & \xrightarrow{\iota_{*,+1}} & K_0(\mathcal{F}_{j,k}) \\ {\scriptstyle \Phi_{j,k}}\downarrow & & & & {\scriptstyle \Phi_{j,k}}\downarrow \\ K_0(\mathcal{A}) & & \xrightarrow{\lambda_\eta} & & K_0(\mathcal{A}). \end{array}$$

(ii) *The restriction of γ_ρ^{-1} to $K_0(\mathcal{F}_{j,k})$ makes the following diagram commutative:*

$$\begin{array}{ccccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j,k}) \\ {\scriptstyle \Phi_{j,k}}\downarrow & & & & {\scriptstyle \Phi_{j,k}}\downarrow \\ K_0(\mathcal{A}) & & \xrightarrow{\lambda_\rho} & & K_0(\mathcal{A}). \end{array}$$

Proof. (i) Put $l = j + k$. Take a projection $p \in M_n(\mathcal{A})$ for some $n \in \mathbb{N}$. Since $\mathcal{A} \otimes M_n(\mathbb{C}) = \sum_{i=1}^{m(l)} (E_i^l \otimes 1)(\mathcal{A} \otimes M_n)(E_i^l \otimes 1)$, by putting $p_i^l = (E_i^l \otimes 1)p(E_i^l \otimes 1) \in (E_i^l \otimes 1)(\mathcal{A} \otimes M_n)(E_i^l \otimes 1) = M_n(E_i^l \mathcal{A} E_i^l)$, we have $p = \sum_{i=1}^{m(l)} p_i^l$. Take $\xi(i) = \xi_1(i) \cdots \xi_k(i) \in B_k(\Lambda_\eta), \nu(i) = \nu_1(i) \cdots \nu_j(i) \in B_j(\Lambda_\rho)$ as in (8.1) so that $\rho_{\nu(i)}(\eta_{\xi(i)}(1)) \geq E_i^l$ and put $\bar{\xi}(i) = \xi_2(i) \cdots \xi_k(i)$ so that $\xi(i) = \xi_1(i)\bar{\xi}(i)$. We have

$$\psi_{j,k*}([p]) = \sum_{i=1}^{m(l)} \oplus [(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \in \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i)).$$

As

$$(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n) \leq T_{\xi_1(i)} T_{\xi_1(i)}^* \otimes 1_n,$$

by the preceding lemma we have

$$T_{\xi_1(i)}^* T_{\xi(i)} S_{\nu(i)} E_i^l = T_{\bar{\xi}(i)} S_{\nu(i)} E_i^l$$

so that by Lemma 7.6

$$\gamma_\eta^{-1}([(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] = [(T_{\bar{\xi}(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\bar{\xi}(i)}^* \otimes 1_n)].$$

Hence $K_0(\mathcal{F}_{j,k})$ goes to $K_0(\mathcal{F}_{j,k-1})$ by the homomorphism γ_η^{-1}. Take $\mu(i) \in B_j(\Lambda_\rho), \bar{\zeta}(i) \in B_{k-1}(\Lambda_\eta)$ such that $T_{\bar{\xi}(i)} S_{\nu(i)} = S_{\mu(i)} T_{\bar{\zeta}(i)}$ for $i = 1, \dots, m(l)$. The element

$$\begin{aligned} & \sum_{i=1}^{m(l)} [(T_{\bar{\xi}(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\bar{\xi}(i)}^* \otimes 1_n)] \\ = & \sum_{i=1}^{m(l)} [(S_{\mu(i)} T_{\bar{\zeta}(i)} \otimes 1_n) p_i^l (T_{\bar{\zeta}(i)}^* S_{\mu(i)}^* \otimes 1_n)] \in K_0(\mathcal{F}_{j,k-1}) \end{aligned}$$

goes to

$$\sum_{i=1}^{m(l)} \sum_{a \in \Sigma^\eta} [(S_{\mu(i)} T_{\bar{\zeta}(i)a} \otimes 1_n)(T_a^* \otimes 1_n) p_i^l (T_a \otimes 1_n)(T_{\bar{\zeta}(i)a}^* S_{\mu(i)}^* \otimes 1_n)] \in K_0(\mathcal{F}_{j,k})$$

by $\iota_{*,+1}$. The element is expressed as

$$\sum_{h=1}^{m(l)}{}^{\oplus}\sum_{i=1}^{m(l)}\sum_{a\in\Sigma^\eta}[(S_{\mu(i)}T_{\bar{\zeta}(i)a}\otimes 1_n)E_h^l(T_a^*\otimes 1_n)p_i^l(T_a\otimes 1_n)E_h^l(T_{\bar{\zeta}(i)a}^*S_{\mu(i)}^*\otimes 1_n)] \tag{8.2}$$

in $\bigoplus_{h=1}^{m(l)}K_0(\mathcal{F}_{j,k}(h))$.

On the other hand, we have

$$\begin{aligned}\lambda_\eta([p]) &= \sum_{a\in\Sigma^\eta}[(T_a^*\otimes 1_n)p(T_a\otimes 1_n)]\\ &= \sum_{h=1}^{m(l)}{}^{\oplus}\sum_{a\in\Sigma^\eta}[E_h^l(T_a^*\otimes 1_n)p(T_a\otimes 1_n)E_h^l]\in\bigoplus_{h=1}^{m(l)}K_0(E_h^l\mathcal{A}E_h^l),\end{aligned}$$

which is expressed as

$$\begin{aligned}&\sum_{h=1}^{m(l)}{}^{\oplus}\sum_{a\in\Sigma^\eta}[(T_{\xi(h)}S_{\nu(h)}E_h^l\otimes 1_n)(T_a^*\otimes 1_n)p(T_a\otimes 1_n)(E_h^lS_{\nu(h)}^*T_{\xi(h)}^*\otimes 1_n)]\\ =&\sum_{h=1}^{m(l)}{}^{\oplus}\sum_{a\in\Sigma^\eta}\sum_{i=1}^{m(l)}[(T_{\xi(h)}S_{\nu(h)}E_h^l\otimes 1_n)(T_a^*\otimes 1_n)p_i^l(T_a\otimes 1_n)(E_h^lS_{\nu(h)}^*T_{\xi(h)}^*\otimes 1_n)]\end{aligned}$$

in $\bigoplus_{h=1}^{m(l)}K_0(\mathcal{F}_{j,k}(h))$. Take $\mu'(h)\in B_j(\Lambda_\rho),\zeta'(h)\in B_k(\Lambda_\eta)$ such that $T_{\xi(h)}S_{\nu(h)} = S_{\mu'(h)}T_{\zeta'(h)}$ so that the above element is

$$\sum_{h=1}^{m(l)}{}^{\oplus}\sum_{i=1}^{m(l)}\sum_{a\in\Sigma^\eta}[(S_{\mu'(h)}T_{\zeta'(h)}E_h^l\otimes 1_n)(T_a^*\otimes 1_n)p_i^l(T_a\otimes 1_n)(E_h^lT_{\zeta'(h)}^*S_{\nu'(h)}^*\otimes 1_n)] \tag{8.3}$$

in $\bigoplus_{h=1}^{m(l)}K_0(\mathcal{F}_{j,k}(h))$. Since for $h,i=1,\dots,m(l),a\in\Sigma^\eta$ the classes of the K-groups coincide such as

$$\begin{aligned}&[(S_{\mu(i)}T_{\bar{\zeta}(i)a}\otimes 1_n)E_h^l(T_a^*\otimes 1_n)p_i^l(T_a\otimes 1_n)E_h^l(T_{\bar{\zeta}(i)a}^*S_{\mu(i)}^*\otimes 1_n)]\\ =&[(S_{\mu'(h)}T_{\zeta'(h)}E_h^l\otimes 1_n)(T_a^*\otimes 1_n)p_i^l(T_a\otimes 1_n)(E_h^lT_{\zeta'(h)}^*S_{\nu'(h)}^*\otimes 1_n)]\in K_0(\mathcal{F}_{j,k}(h)),\end{aligned}$$

the element of (8.2) is equal to the element of (8.3) in $K_0(\mathcal{F}_{j,k})$. Thus (i) holds.

(ii) is similar to (i). □

We note that for $j,k\in\mathbb{Z}_+$,

$$\begin{aligned}K_0(\mathcal{F}_{\rho,k}) &= \lim_j\{\iota_{+1,*}:K_0(\mathcal{F}_{j,k})\longrightarrow K_0(\mathcal{F}_{j+1,k})\},\\ K_0(\mathcal{F}_{j,\eta}) &= \lim_k\{\iota_{*,+1}:K_0(\mathcal{F}_{j,k})\longrightarrow K_0(\mathcal{F}_{j,k+1})\}.\end{aligned}$$

The following lemma is direct.

Lemma 8.8. *For $k,j\in\mathbb{Z}_+$, the following diagrams are commutative:*

(i)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{j,k-1})\\ {\scriptstyle\iota_{+1,*}}\downarrow & & {\scriptstyle\iota_{+1,*}}\downarrow\\ K_0(\mathcal{F}_{j+1,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{j+1,k-1}).\end{array}$$

Hence γ_η^{-1} yields a homomorphism from $K_0(\mathcal{F}_{\rho,k})$ to $K_0(\mathcal{F}_{\rho,k-1})$.

(ii)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,k}) \\ {\scriptstyle \iota_{*,+1}}\downarrow & & {\scriptstyle \iota_{*,+1}}\downarrow \\ K_0(\mathcal{F}_{j,k+1}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,k+1}). \end{array}$$

Hence γ_ρ^{-1} yields a homomorphism from $K_0(\mathcal{F}_{j,\eta})$ to $K_0(\mathcal{F}_{j-1,\eta})$.

The homomorphisms

$$\iota_{+1,*} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j+1,k}), \qquad \iota_{*,+1} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j,k+1})$$

are naturally induce homomorphisms

$$K_0(\mathcal{F}_{j,\eta}) \longrightarrow K_0(\mathcal{F}_{j+1,\eta}), \qquad \iota_{*,+1} : K_0(\mathcal{F}_{\rho,k}) \longrightarrow K_0(\mathcal{F}_{\rho,k+1})$$

which we denote by $\iota_{+1,\eta}, \iota_{\rho,+1}$ respectively. They are also induced by the identities (5.1), (5.2) respectively.

Lemma 8.9. *For $k, j \in \mathbb{Z}_+$, the following diagrams are commutative:*

(i)

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,k-1}) \\ {\scriptstyle \iota_{\rho,+1}}\downarrow & & {\scriptstyle \iota_{\rho,+1}}\downarrow \\ K_0(\mathcal{F}_{\rho,k+1}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,k}). \end{array}$$

(ii)

$$\begin{array}{ccc} K_0(\mathcal{F}_{j,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,\eta}) \\ {\scriptstyle \iota_{+1,\eta}}\downarrow & & {\scriptstyle \iota_{+1,\eta}}\downarrow \\ K_0(\mathcal{F}_{j+1,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j,\eta}). \end{array}$$

Proof. (i) As in the proof of Lemma 8.8, one may take an element of $K_0(\mathcal{F}_{\rho,k})$ as in the following form:

$$\sum_{i=1}^{m(l)}{}^{\oplus}[(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \in \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i))$$

for some projection $p \in M_n(\mathcal{A})$ and j, l with $l = j+k$, where $p_i^l = (E_i^l \otimes 1)p(E_i^l \otimes 1) \in (E_i^l \otimes 1)(\mathcal{A} \otimes M_n)(E_i^l \otimes 1) = M_n(E_i^l \mathcal{A} E_i^l)$. Let $\xi(i) = \xi_1(i)\bar{\xi}(i)$ with $\xi_1(i) \in \Sigma^\eta$, $\bar{\xi}(i) \in B_{k-1}(\Lambda_\eta)$. One may assume that $T_{\xi(i)}S_{\nu(i)} \neq 0$ so that $T_{\bar{\xi}(i)}S_{\nu(i)} = S_{\nu(i)'}T_{\bar{\xi}(i)'}$ for some $\nu(i)' \in B_j(\Lambda_\rho), \bar{\xi}(i)' \in B_{k-1}(\Lambda_\eta)$. As in the proof of Lemma 8.8, one has

$$\begin{aligned} \gamma_\eta^{-1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] &= [(T_{\bar{\xi}(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\bar{\xi}(i)}^* \otimes 1_n)] \\ &= [(S_{\nu(i)'}T_{\bar{\xi}(i)'} \otimes 1_n)p_i^l(S_{\nu(i)'}^* T_{\bar{\xi}(i)'}^* \otimes 1_n)] \end{aligned}$$

Hence we have

$$
\begin{aligned}
&\iota_{*,+1} \circ \gamma_\eta^{-1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)]\\
=&\iota_{*,+1}([S_{\nu(i)'}T_{\bar{\xi}(i)'} \otimes 1_n)p_i^l(T_{\bar{\xi}(i)'}^* S_{\nu(i)'}^* \otimes 1_n])\\
=&\sum_{b\in\Sigma^\eta}[(S_{\nu(i)'}T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* \otimes 1_n)]
\end{aligned}
$$

On the other hand, we have $T_{\xi(i)}S_{\nu(i)} = T_{\xi(i)_1}S_{\nu(i)'}T_{\bar{\xi}(i)'}$ so that

$$
\begin{aligned}
&\iota_{*,+1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)]\\
=&\sum_{b\in\Sigma^\eta}[(T_{\xi(i)_1}S_{\nu(i)'}T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* T_{\xi(i)_1}^* \otimes 1_n)]
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\gamma_\eta^{-1} \circ \iota_{*,+1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)]\\
=&\sum_{b\in\Sigma^\eta}\gamma_\eta^{-1}([(T_{\xi(i)_1}S_{\nu(i)'}T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* T_{\xi(i)_1}^* \otimes 1_n)])\\
=&\sum_{b\in\Sigma^\eta}[(S_{\nu(i)'}T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n)p_i^l(T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* \otimes 1_n)].
\end{aligned}
$$

(ii) The proof is completely symmetric to the above proof. □

Since the homomorphisms $\lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ are mutuallu commutative, the map λ_η induces a homomorphism on the inductive limit $G_\rho = \lim\{\lambda_\rho : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}$ and similarly the map λ_ρ does on the inductive limit G_η. They are still denote by $\lambda_\rho, \lambda_\eta$ respectively.

Lemma 8.10. *For $k, j \in \mathbb{Z}_+$, the following diagrams are commutative:*

(i)

$$
\begin{array}{ccccc}
K_0(\mathcal{F}_{\rho,k}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,k-1}) & \xrightarrow{\iota_{\rho,+1}} & K_0(\mathcal{F}_{\rho,k}) \\
{\scriptstyle \Phi_{\rho,k}}\downarrow & & & & {\scriptstyle \Phi_{\rho,k}}\downarrow \\
G_\rho & & \xrightarrow{\lambda_\eta} & & G_\rho.
\end{array}
$$

(ii)

$$
\begin{array}{ccccc}
K_0(\mathcal{F}_{j,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{j-1,\eta}) & \xrightarrow{\iota_{+1,\eta}} & K_0(\mathcal{F}_{j,\eta}) \\
{\scriptstyle \Phi_{j,\eta}}\downarrow & & & & {\scriptstyle \Phi_{j,\eta}}\downarrow \\
G_\eta & & \xrightarrow{\lambda_\rho} & & G_\eta.
\end{array}
$$

Proof. (i) As in the proof of Lemma 8.7 and Lemma 8.9 one may take an element of $K_0(\mathcal{F}_{\rho,k})$ as in the following form:

$$
\sum_{i=1}^{m(l)}{}^{\oplus}[(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \in \bigoplus_{i=1}^{m(l)} K_0(\mathcal{F}_{j,k}(i))
$$

for some projection $p \in M_n(\mathcal{A})$ and j, l with $l = j+k$, where $p_i^l = (E_i^l \otimes 1)p(E_i^l \otimes 1)$. Keep the notations as in the proof of Lemma 8.7, we have

$$\begin{aligned}
&\iota_{*,+1} \circ \gamma_\eta^{-1}([(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)]) \\
=& \sum_{b \in \Sigma^\eta} [(S_{\nu(i)'} T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n) p_i^l (T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* \otimes 1_n)]
\end{aligned}$$

so that

$$\begin{aligned}
&\Phi_{\rho,k} \circ \iota_{*,+1} \circ \gamma_\eta^{-1}([(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)] \\
=& \sum_{b \in \Sigma^\eta} \Phi_{\rho,k} \circ ([S_{\nu(i)'} T_{\bar{\xi}(i)'b} \otimes 1_n)(T_b^* \otimes 1_n) p_i^l (T_b \otimes 1_n)(T_{\bar{\xi}(i)'b}^* S_{\nu(i)'}^* \otimes 1_n])) \\
=& \sum_{b \in \Sigma^\eta} [(T_b^* \otimes 1_n) p_i^l (T_b \otimes 1_n)) \\
=& \lambda_\eta([p_i^l]) = \lambda_\eta \circ \Phi_{\rho,k}([(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i^l (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n)]).
\end{aligned}$$

Therefore we have $\Phi_{\rho,k} \circ \iota_{\rho,+1} \circ \gamma_\eta^{-1} = \lambda_\eta \circ \Phi_{\rho,k}$.

(ii) The proof is completely symmetric to the above proof. □

Put for $j, k \in \mathbb{Z}_+$

$$\begin{aligned}
G_{\rho,k} &= K_0(\mathcal{F}_{\rho,k})(\cong G_\rho = \lim\{\lambda_\rho : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}), \\
G_{j,\eta} &= K_0(\mathcal{F}_{j,\eta})(\cong G_\eta = \lim\{\lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})\}).
\end{aligned}$$

The map $\lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ naturally gives rise to a homomorphism from $G_{\rho,k}$ to $G_{\rho,k+1}$ which we will still denote by λ_η. Similarly we have a homomorphism λ_ρ from $G_{j,\eta}$ to $G_{j+1,\eta}$.

Lemma 8.11. *For $k, j \in \mathbb{Z}_+$, the following diagrams are commutative:*

(i)

$$\begin{array}{ccc}
K_0(\mathcal{F}_{\rho,k}) & \xrightarrow{\iota_{\rho,+1}} & K_0(\mathcal{F}_{\rho,k+1}) \\
\| & & \| \\
G_{\rho,k} & \xrightarrow{\lambda_\eta} & G_{\rho,k+1}.
\end{array}$$

(ii)

$$\begin{array}{ccc}
K_0(\mathcal{F}_{j,\eta}) & \xrightarrow{\iota_{+1,\eta}} & K_0(\mathcal{F}_{j+1,\eta}) \\
\| & & \| \\
G_{j,\eta} & \xrightarrow{\lambda_\rho} & G_{j+1,\eta}.
\end{array}$$

We denote the abelian group $K_0(\mathcal{F}_{\rho,\eta})$ by $G_{\rho,\eta}$. Since

$$\begin{aligned}
K_0(\mathcal{F}_{\rho,\eta}) &= \lim_k \{\iota_{rho,+1} : K_0(\mathcal{F}_{\rho,k}) \longrightarrow K_0(\mathcal{F}_{\rho,k+1})\} \\
&= \lim_j \{\iota_{+1,eta} : K_0(\mathcal{F}_{j,\eta}) \longrightarrow K_0(\mathcal{F}_{j+1,\eta})\},
\end{aligned}$$

one has

$$\begin{aligned}
G_{\rho,\eta} &= \lim_k \{\lambda_\eta : G_{\rho,k} \longrightarrow G_{\rho,k+1}\} \\
&= \lim_j \{\lambda_\rho : G_{j,\eta} \longrightarrow G_{j+1,\eta}\}.
\end{aligned}$$

Define two endomorphisms

$$\sigma_\eta \text{ on } G_{\rho,\eta} = \lim_k \{\lambda_\eta : G_{\rho,k} \longrightarrow G_{\rho,k+1}\} \quad \text{and}$$
$$\sigma_\rho \text{ on } G_{\rho,\eta} = \lim_j \{\lambda_\rho : G_{j,\eta} \longrightarrow G_{j+1,\eta}\}$$

by setting

$$\sigma_\rho : [g,k] \in G_{\rho,k} \longrightarrow [g,k-1] \in G_{\rho,k-1} \text{ for } g \in G_\rho \text{ and}$$
$$\sigma_\eta : [h,j] \in G_{j,\eta} \longrightarrow [h,j-1] \in G_{j-1,\eta} \text{ for } h \in G_\eta.$$

Therefore we have

Lemma 8.12.

(i) *There exists an isomorphism* $\Phi_{\rho,\infty} : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow G_{\rho,\eta}$ *such that the following diagrams are commutative:*

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ {\scriptstyle \Phi_{\rho,\infty}}\downarrow & & {\scriptstyle \Phi_{\rho,\infty}}\downarrow \\ G_{\rho,\eta} & \xrightarrow{\sigma_\eta} & G_{\rho,\eta} \end{array}$$

and hence

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{id-\gamma_\eta^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ {\scriptstyle \Phi_{\rho,\infty}}\downarrow & & {\scriptstyle \Phi_{\rho,\infty}}\downarrow \\ G_{\rho,\eta} & \xrightarrow{id-\sigma_\eta} & G_{\rho,\eta}. \end{array}$$

(ii) *There exists an isomorphism* $\Phi_{\infty,\eta} : K_0(\mathcal{F}_{\rho,\eta}) \longrightarrow G_{\rho,\eta}$ *such that the following diagrams are commutative:*

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\gamma_\rho^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ {\scriptstyle \Phi_{\infty,\eta}}\downarrow & & {\scriptstyle \Phi_{\infty,\eta}}\downarrow \\ G_{\rho,\eta} & \xrightarrow{\sigma_\rho} & G_{\rho,\eta} \end{array}$$

and hence

$$\begin{array}{ccc} K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{id-\gamma_\rho^{-1}} & K_0(\mathcal{F}_{\rho,\eta}) \\ {\scriptstyle \Phi_{\infty,\eta}}\downarrow & & {\scriptstyle \Phi_{\infty,\eta}}\downarrow \\ G_{\rho,\eta} & \xrightarrow{id-\sigma_\rho} & G_{\rho,\eta}. \end{array}$$

Let us denote by $J_\mathcal{A}$ the natural embedding $\mathcal{A} = \mathcal{F}_{0,0} \hookrightarrow \mathcal{F}_{\rho,\eta}$. There exists a homomorphism

$$J_{\mathcal{A}*} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{F}_{\rho,\eta}).$$

Lemma 8.13. *The homomorphism* $J_{\mathcal{A}*} : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{F}_{\rho,\eta})$ *is injective such that*

$$J_{\mathcal{A}*} \circ \lambda_\rho = \gamma_\rho^{-1} \circ J_{\mathcal{A}*} \quad \textit{and} \quad J_{\mathcal{A}*} \circ \lambda_\eta = \gamma_\eta^{-1} \circ J_{\mathcal{A}*}.$$

Proof. We will first show that the endomorphisms $\lambda_\rho, \lambda_\eta$ on $K_0(\mathcal{A})$ are both injective. Put a projection $Q_\alpha = S_\alpha S_\alpha^*$ and a subalgebra $\mathcal{A}_\alpha = \rho_\alpha(\mathcal{A})$ of $\mathcal{A}$ for $\alpha \in \Sigma^\rho$. Then the endomorphism ρ_α on $\mathcal{A}$ extends to an isomorphism from $\mathcal{A}Q_\alpha$ onto $\mathcal{A}_\alpha$ by setting $\rho_\alpha(x) = S_\alpha^* x S_\alpha, x \in \mathcal{A}Q_\alpha$ whose inverse is $\phi_\alpha : \mathcal{A}_\alpha \longrightarrow \mathcal{A}Q_\alpha$ defined by $\phi_\alpha(y) = S_\alpha y S_\alpha^*, y \in \mathcal{A}_\alpha$. Hence the induced homomorphism $\rho_{\alpha *} : K_0(\mathcal{A}Q_\alpha) \longrightarrow K_0(\mathcal{A}_\alpha)$ is an isomorphism. Since $\mathcal{A} = \bigoplus_{\alpha\in\Sigma^\rho} Q_\alpha \mathcal{A}$, the homomorphism

$$\sum_{\alpha\in\Sigma^\rho} \phi_{\alpha *} \circ \rho_{\alpha *} : K_0(\mathcal{A}) \longrightarrow \bigoplus_{\alpha\in\Sigma^\rho} K_0(Q_\alpha \mathcal{A})$$

is an isomorphism, one may identify $K_0(\mathcal{A}) = \bigoplus_{\alpha\in\Sigma^\rho} K_0(Q_\alpha\mathcal{A})$. Let $g \in K_0(\mathcal{A})$ satisfy $\lambda_\rho(g) = 0$. Put $g_\alpha = \phi_{\alpha *} \circ \rho_{\alpha *}(g) \in K_0(Q_\alpha\mathcal{A})$ for $\alpha \in \Sigma^\rho$ so that $g = \sum_{\alpha\in\Sigma^\rho} g_\alpha$. As $\rho_{\beta *} \circ \phi_{\alpha *} = 0$ for $\beta \neq \alpha$, one sees $\rho_{\beta *}(g_\alpha) = 0$ for $\beta \neq \alpha$. Hence

$$0 = \lambda_\rho(g) = \sum_{\beta\in\Sigma^\rho} \sum_{\alpha\in\Sigma^\rho} \rho_{\beta *}(g_\alpha) = \sum_{\alpha\in\Sigma^\rho} \rho_{\alpha *}(g_\alpha) \in \bigoplus_{\alpha\in\Sigma^\rho} K_0(\mathcal{A}_\alpha).$$

It follows that $\rho_{\alpha *}(g_\alpha) = 0$ in $K_0(\mathcal{A}_\alpha)$. Since $\rho_{\alpha *} : K_0(Q_\alpha\mathcal{A}) \longrightarrow K_0(\mathcal{A}_\alpha)$ is isomorphic, one sees that $g_\alpha = 0$ in $K_0(\mathcal{A}Q_\alpha)$ for all $\alpha \in \Sigma^\rho$. This implies that $g = \sum_{\alpha\in\Sigma^\rho} g_\alpha = 0$ in $K_0(\mathcal{A})$. Therefore the endomorphism λ_ρ on $K_0(\mathcal{A})$ is injective, and similarly so is λ_η.

By the previous lemma, there exists an isomorphism $\Phi_{j,k} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{A})$ such that the diagram

$$\begin{array}{ccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\iota_{+1,*}} & K_0(\mathcal{F}_{j+1,k}) \\
{\scriptstyle \Phi_{j,k}}\downarrow & & {\scriptstyle \Phi_{j+1,k}}\downarrow \\
K_0(\mathcal{A}) & \xrightarrow{\lambda_\rho} & K_0(\mathcal{A})
\end{array}$$

is commutative so that the embedding $\iota_{+1,*} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j+1,k})$ is injective, and similarly $\iota_{*,+1} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j,k+1})$ is injective. Hence for $n, m \in \mathbb{N}$, the homomorphism

$$\iota_{n,m} : K_0(\mathcal{A}) = K_0(\mathcal{F}_{0,0}) \longrightarrow K_0(\mathcal{F}_{n,m})$$

defined by the compositions of $\iota_{+1,*}$ and $\iota_{*,+1}$ is injective. By [38, Theorem 6.3.2 (iii)], one knows $\mathrm{Ker}(J_{\mathcal{A}*}) = \cup_{n,m\in\mathbb{N}} \mathrm{Ker}(\iota_{n,m})$, so that $\mathrm{Ker}(J_{\mathcal{A}*}) = 0$. $\square$

We henceforth identify the group $K_0(\mathcal{A})$ with its image $J_{\mathcal{A}*}(K_0(\mathcal{A}))$ in $K_0(\mathcal{F}_{\rho,\eta})$. As in the above proof, not only $K_0(\mathcal{A})(= K_0(\mathcal{F}_{0,0}))$ but also the groups $K_0(\mathcal{F}_{j,k})$ for j, k are identified with subgroups of $K_0(\mathcal{F}_{\rho,\eta})$ via injective homomorphisms from $K_0(\mathcal{F}_{j,k})$ to $K_0(\mathcal{F}_{\rho,\eta})$ induced by the embeddings of $\mathcal{F}_{j,k}$ into $\mathcal{F}_{\rho,\eta}$.

We note that

$$(\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta}) = (\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta}),$$
$$(\mathrm{id} - \gamma_\rho)K_0(\mathcal{F}_{\rho,\eta}) = (\mathrm{id} - \gamma_\rho^{-1})K_0(\mathcal{F}_{\rho,\eta})$$

and

$$(\mathrm{id} - \gamma_\rho) \cap (\mathrm{id} - \gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta}) = (\mathrm{id} - \gamma_\rho^{-1}) \cap (\mathrm{id} - \gamma_\eta^{-1}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}).$$

Denote by $(\mathrm{id}-\gamma_\rho)K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id}-\gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$ the subgroup of $K_0(\mathcal{F}_{\rho,\eta})$ generated by $(\mathrm{id} - \gamma_\rho)K_0(\mathcal{F}_{\rho,\eta})$ and $(\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$.

Lemma 8.14. *An element in $K_0(\mathcal{F}_{\rho,\eta})$ is equivalent to an element of $K_0(\mathcal{A})$ modulo the subgroup $(\mathrm{id} - \gamma_\rho)K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$.*

Proof. For $g \in K_0(\mathcal{F}_{\rho,\eta})$, we may assume that $g \in K_0(\mathcal{F}_{j,k})$ for some $j,k \in \mathbb{Z}_+$. As γ_ρ^{-1} commutes with γ_η^{-1}, one sees that $(\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) \in K_0(\mathcal{A})$. Put $g_1 = \gamma_\rho^{-1}(g)$ so that

$$g - (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) = g - \gamma_\rho^{-1}(g) + g_1 - (\gamma_\rho^{-1})^{j-1} \circ (\gamma_\eta^{-1})^k(g_1).$$

We inductively see that $g - (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g)$ belongs to the subgroup $(\mathrm{id} - \gamma_\rho)K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$. Hence g is equivalent to $(\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g)$ modulo $(\mathrm{id} - \gamma_\rho)K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$. $\square$

Denote by $(\mathrm{id} - \lambda_\rho)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})$ the subgroup of $K_0(\mathcal{A})$ generated by $(\mathrm{id} - \lambda_\rho)K_0(\mathcal{A})$ and $(\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})$.

Lemma 8.15. *For $g \in K_0(\mathcal{A})$, the condition $g \in (\mathrm{id} - \gamma_\rho^{-1})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$ implies $g \in (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\eta)K_0(\mathcal{A})$.*

Proof. By the assumption that $g \in (\mathrm{id} - \gamma_\rho^{-1})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$, there exist $h_1, h_2 \in K_0(\mathcal{F}_{\rho,\eta})$ such that $g = (\mathrm{id} - \gamma_\rho^{-1})(h_1) + (\mathrm{id} - \gamma_\eta^{-1})(h_2)$. We may assume that $h_1, h_2 \in K_0(\mathcal{F}_{j,k})$ for large enough $j,k \in \mathbb{Z}_+$. Put $e_i = (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(h_i)$ which belongs to $K_0(\mathcal{F}_{0,0})(= K_0(\mathcal{A}))$ for $i = 0,1$. It follows that

$$\lambda_\rho^j \circ \lambda_\eta^k(g) = (\mathrm{id} - \lambda_\eta)(e_1) + (\mathrm{id} - \lambda_\rho)(e_2).$$

Now $g \in K_0(\mathcal{A})$ and $\lambda_\rho^j \circ \lambda_\eta^k(g) \in (\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A}) \subset K_0(\mathcal{A})$. As in the proof of the preceding lemma, by putting $g^{(n)} = \lambda_\rho^n(g), g^{(n,m)} = \lambda_\eta^m(g^{(n)}) \in K_0(\mathcal{A})$ we have

$$\begin{aligned}
& g - \lambda_\rho^j \circ \lambda_\eta^k(g) \\
=& g - \lambda_\rho(g) + g^{(1)} - \lambda_\rho(g^{(1)}) + g^{(2)} - \lambda_\rho(g^{(2)}) + \cdots + g^{(j-1)} - \lambda_\rho(g^{(j-1)}) \\
& + g^{(j)} - \lambda_\eta(g^{(j)}) + g^{(j,1)} - \lambda_\eta(g^{(j,1)}) + g^{(j,2)} - \lambda_\eta(g^{(j,2)}) + \cdots \\
& + g^{(j,k-1)} - \lambda_\eta(g^{(j,k-1)}) \\
=& (\mathrm{id} - \lambda_\rho)(g + g^{(1)} + \cdots + g^{(j-1)}) + (\mathrm{id} - \lambda_\eta)(g^{(j)} + g^{(j,1)} + \cdots + g^{(j,k-1)})
\end{aligned}$$

Since $\lambda_\rho^j \circ \lambda_\eta^k(g) \in (\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A})$ and

$$\begin{aligned}
&(\mathrm{id} - \lambda_\rho)(g + g^{(1)} + \cdots + g^{(j-1)}) \in (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A}), \\
&(\mathrm{id} - \lambda_\eta)(g^{(j)} + g^{(j,1)} + \cdots + g^{(j,k-1)}) \in (\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}),
\end{aligned}$$

we have

$$g \in (\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A}).$$

$\square$

Hence we obtain the following lemma for the cokernel.

Lemma 8.16. *The quotient group*

$$K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id} - \gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id} - \gamma_\rho^{-1})K_0(\mathcal{F}_{\rho,\eta}))$$

is isomorphic to the quotient group

$$K_0(\mathcal{A})/((\mathrm{id} - \lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id} - \lambda_\rho)K_0(\mathcal{A})).$$

Proof. Surjectivity of the quotient map

$$K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{F}_{\rho,\eta})/((\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta}) + (\mathrm{id}-\gamma_\rho^{-1})K_0(\mathcal{F}_{\rho,\eta}))$$

comes from Lemma 8.14. Its kernel coincides with

$$(\mathrm{id}-\lambda_\eta)K_0(\mathcal{A}) + (\mathrm{id}-\lambda_\rho)K_0(\mathcal{A})$$

by the preceing lemma. Hence we have a desired assertion. □

For the kernel, we have

Lemma 8.17. *The subgroup*

$$\mathrm{Ker}(\mathrm{id}-\gamma_\eta^{-1}) \cap \mathrm{Ker}(\mathrm{id}-\gamma_\rho^{-1}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})$$

is isomorphic to the subgroup

$$\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \cap \mathrm{Ker}(\mathrm{id}-\lambda_\rho) \text{ in } K_0(\mathcal{A})$$

through $J_{\mathcal{A}*}$.

Proof. For $g \in \mathrm{Ker}(\mathrm{id}-\gamma_\eta^{-1}) \cap \mathrm{Ker}(\mathrm{id}-\gamma_\rho^{-1})$ in $K_0(\mathcal{F}_{\rho,\eta})$, one may assume that $g \in K_0(\mathcal{F}_{j,k})$ for some $j,k \in \mathbb{Z}_+$ so that $(\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) \in K_0(\mathcal{A})$. Through the identification between $J_{\mathcal{A}*}(K_0(\mathcal{A}))$ and $K_0(\mathcal{A})$ via $J_{\mathcal{A}*}$, one may assume that $\lambda_\eta = \gamma_\eta^{-1}$ and $\lambda_\rho = \gamma_\rho^{-1}$ on $K_0(\mathcal{A})$. As $g \in \mathrm{Ker}(\mathrm{id}-\gamma_\eta^{-1}) \cap \mathrm{Ker}(\mathrm{id}-\gamma_\rho^{-1})$, one has $g = (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) \in K_0(\mathcal{A})$. This implies that $g \in \mathrm{Ker}(\mathrm{id}-\lambda_\eta) \cap \mathrm{Ker}(\mathrm{id}-\lambda_\rho)$ in $K_0(\mathcal{A})$. The converse inclusion relation $\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \cap \mathrm{Ker}(\mathrm{id}-\lambda_\rho) \subset \mathrm{Ker}(\mathrm{id}-\gamma_\eta^{-1}) \cap \mathrm{Ker}(\mathrm{id}-\gamma_\rho^{-1})$ is clear through the above identification. □

Therefore we have

Proposition 8.18. *There exists a short exact sequence:*

$$\begin{aligned} 0 &\longrightarrow K_0(\mathcal{A})/((\mathrm{id}-\lambda_{\eta*})K_0(\mathcal{A}) + (\mathrm{id}-\lambda_{\rho*})K_0(\mathcal{A})) \\ &\longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta}) \\ &\longrightarrow \mathrm{Ker}(\mathrm{id}-\lambda_{\eta*}) \cap \mathrm{Ker}(\mathrm{id}-\lambda_{\rho*}) \text{ in } K_0(\mathcal{A}) \\ &\longrightarrow 0. \end{aligned}$$

Let $\mathcal{F}_\rho$ be the fixed point algebra $(\mathcal{O}_\rho)^{\hat\rho}$ of the C^*-algebra $\mathcal{O}_\rho$ by the gauge action $\hat\rho$ for the C^*-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma^\rho)$. The algebra $\mathcal{F}_\rho$ is isomorphic to the subalgebra $\mathcal{F}_{\rho,0}$ of $\mathcal{F}_{\rho,\eta}$ in a natural way. As in the proof of Lemma 8.14, the group $K_0(\mathcal{F}_{\rho,0})$ is regarded as a subgroup of $K_0(\mathcal{F}_{\rho,\eta})$ and the restriction of γ_η^{-1} to $K_0(\mathcal{F}_{\rho,0})$ satisfies $\gamma_\eta^{-1}(K_0(\mathcal{F}_{\rho,0})) \subset K_0(\mathcal{F}_{\rho,0})$ so that γ_η^{-1} yields an endomorphism on $K_0(\mathcal{F}_\rho)$, which we still denote by γ_η^{-1}.

For the group $K_1(\mathcal{O}^\kappa_{\rho,\eta})$, we provide several lemmas.

Lemma 8.19.

(i) *An element in $K_0(\mathcal{F}_{\rho,\eta})$ is equivalent to an element of $K_0(\mathcal{F}_{\rho,0})(= K_0(\mathcal{F}_\rho))$ modulo the subgroup* $(\mathrm{id}-\gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$.

(ii) *If $g \in K_0(\mathcal{F}_{\rho,0})(= K_0(\mathcal{F}_\rho))$ belongs to $(\mathrm{id}-\gamma_\eta)K_0(\mathcal{F}_{\rho,\eta})$, then g belongs to* $(\mathrm{id}-\gamma_\eta)K_0(\mathcal{F}_\rho)$.

As γ_ρ commutes with γ_η on $K_0(\mathcal{F}_{\rho,\eta})$, it naturally acts on the quotient group $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$. We denote it by $\bar\gamma_\rho$. Similarly λ_ρ naturally induces an endomorphism on $K_0(\mathcal{A})/(\mathrm{id}-\lambda_\eta)K_0(\mathcal{A})$. We denote it by $\bar\lambda_\rho$.

Lemma 8.20. (i) *The quotient group* $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$ *is isomorphic to the quotient group* $K_0(\mathcal{F}_\rho)/(\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_\rho)$*, that is also isomorphic to the quotient group* $K_0(\mathcal{A})/(1-\lambda_\eta)K_0(\mathcal{A})$.

(ii) *The kernel of* $\mathrm{id}-\bar{\gamma}_\rho$ *in* $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$ *is isomorphic to the kernel of* $\mathrm{id}-\bar{\lambda}_\rho$ *in* $K_0(\mathcal{A})/(\mathrm{id}-\lambda_\eta)K_0(\mathcal{A})$*. That is*

$$\begin{aligned}&\mathrm{Ker}(\mathrm{id}-\bar{\gamma}_\rho) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})\\ \cong&\mathrm{Ker}(\mathrm{id}-\bar{\lambda}_\rho) \text{ in } K_0(\mathcal{A})/(\mathrm{id}-\lambda_\eta)K_0(\mathcal{A}).\end{aligned}$$

Proof. (i) The fact that the three quotient groups

$$K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta}),\quad K_0(\mathcal{F}_\rho)/(\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_\rho),\quad K_0(\mathcal{A})/(\mathrm{id}-\lambda_\eta)K_0(\mathcal{A})$$

are naturally isomorphic is similarly proved to the previous discussions. (ii) The kernel $\mathrm{Ker}(\mathrm{id}-\bar{\gamma}_\rho)$ in $K_0(\mathcal{F}_{\rho,\eta})/(\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_{\rho,\eta})$ is isomorphic to the kernel $\mathrm{Ker}(\mathrm{id}-\bar{\gamma}_\rho)$ in $K_0(\mathcal{F}_\rho)/(\mathrm{id}-\gamma_\eta^{-1})K_0(\mathcal{F}_\rho)$ which is isomorphic to the kernel $\mathrm{Ker}(\mathrm{id}-\bar{\lambda}_\rho)$ in $K_0(\mathcal{A})/(1-\lambda_\eta)K_0(\mathcal{A})$. □

Lemma 8.21. *The kernel of* $\mathrm{id}-\gamma_\rho$ *in* $K_0(\mathcal{F}_{\rho,\eta})$ *is isomorphic to the kernel of* $\mathrm{id}-\gamma_\rho$ *in* $K_0(\mathcal{F}_\rho)$ *that is also isomorphic to the kernel of* $\mathrm{id}-\lambda_\eta$ *in* $K_0(\mathcal{A})$ *such that the quotient group*

$$(\mathrm{Ker}(\mathrm{id}-\gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id}-\gamma_\rho)(\mathrm{Ker}(\mathrm{id}-\gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))$$

is isomorphic to the quotient group

$$(\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \text{ in } K_0(\mathcal{A}))/(\mathrm{id}-\lambda_\rho)(\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \text{ in } K_0(\mathcal{A})).$$

That is

$$\begin{aligned}&(\mathrm{Ker}(\mathrm{id}-\gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))/(\mathrm{id}-\gamma_\rho)(\mathrm{Ker}(\mathrm{id}-\gamma_\eta) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))\\ \cong&(\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \text{ in } K_0(\mathcal{A}))/(\mathrm{id}-\lambda_\rho)(\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \text{ in } K_0(\mathcal{A})).\end{aligned}$$

Proof. The proofs are similar to the previous discussions. □

Therefore we have

Proposition 8.22. *There exists a short exact sequence:*

$$\begin{aligned}0 &\longrightarrow (\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \text{ in } K_0(\mathcal{A}))/(\mathrm{id}-\lambda_\rho)(\mathrm{Ker}(\mathrm{id}-\lambda_\eta) \text{ in } K_0(\mathcal{A}))\\ &\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta})\\ &\longrightarrow \mathrm{Ker}(\mathrm{id}-\bar{\lambda}_\rho) \text{ in } (K_0(\mathcal{A})/(\mathrm{id}-\lambda_\eta)K_0(\mathcal{A}))\\ &\longrightarrow 0.\end{aligned}$$

Consequently we have

Theorem 8.23. *Suppose that a* C^**-textile dynamical system* $(\mathcal{A},\rho,\eta,\Sigma^\rho,\Sigma^\eta,\kappa)$ *forms square. Then there exist short exact sequences for their K-groups as in the following way:*

$$\begin{aligned}0 &\longrightarrow K_0(\mathcal{A})/((\mathrm{id}-\lambda_\eta)K_0(\mathcal{A})+(\mathrm{id}-\lambda_\rho)K_0(\mathcal{A}))\\ &\longrightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta})\\ &\longrightarrow \mathrm{Ker}(\mathrm{id}-\lambda_\eta)\cap\mathrm{Ker}(\mathrm{id}-\lambda_\rho) \text{ in } K_0(\mathcal{A})\\ &\longrightarrow 0\end{aligned}$$

and

$$
\begin{aligned}
0 &\longrightarrow (\operatorname{Ker}(\mathrm{id}-\lambda_\eta) \text{ in } K_0(\mathcal{A}))/(\mathrm{id}-\lambda_\rho)(\operatorname{Ker}(\mathrm{id}-\lambda_\eta) \text{ in } K_0(\mathcal{A}))\\
&\longrightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta})\\
&\longrightarrow \operatorname{Ker}(\mathrm{id}-\bar{\lambda}_\rho) \text{ in } (K_0(\mathcal{A})/(\mathrm{id}-\lambda_\eta)K_0(\mathcal{A}))\\
&\longrightarrow 0
\end{aligned}
$$

where the endomorphisms $\lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$ *are defined by*

$$
\begin{aligned}
\lambda_\rho([p]) &= \sum_{\alpha\in\Sigma^\rho} [\rho_\alpha(p)] \in K_0(\mathcal{A}) \text{ for } [p] \in K_0(\mathcal{A}),\\
\lambda_\eta([p]) &= \sum_{a\in\Sigma^\eta} [\eta_a(p)] \in K_0(\mathcal{A}) \text{ for } [p] \in K_0(\mathcal{A}).
\end{aligned}
$$

9. Examples

1. LR-textile λ-graph systems.

A symbolic matrix $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ is a matrix whose components consist of formal sums of elements of an alphabet Σ, such as

$$
\mathcal{M} = \begin{bmatrix} a & a+c \\ c & 0 \end{bmatrix} \qquad \text{where } \Sigma = \{a,b,c\}.
$$

$\mathcal{M}$ is said to be essential if there is no zero column or zero row. $\mathcal{M}$ is said to be left-resolving if for each column a symbol does not appear in two different rows. For example, $\begin{bmatrix} a & a+b \\ c & 0 \end{bmatrix}$ is left-resolving, but $\begin{bmatrix} a & a+b \\ c & b \end{bmatrix}$ is not left-resolving because of b at the second column. We asssume that symbolic matrices are always essential and left-resolving. We denote by $\Sigma^{\mathcal{M}}$ the alphabet Σ of the symbolic matrix $\mathcal{M}$.

Let $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ and $\mathcal{M}' = [\mathcal{M}'(i,j)]_{i,j=1}^N$ be $N\times N$ symbolic matrices over $\Sigma^{\mathcal{M}}$ and $\Sigma^{\mathcal{M}'}$ respectively. Suppose that there is a bijection $\kappa : \Sigma^{\mathcal{M}} \longrightarrow \Sigma^{\mathcal{M}'}$. Following Nasu's terminology [28] we say that $\mathcal{M}$ and $\mathcal{M}'$ are equivalent under specification κ, or simply a specified equivalence if $\mathcal{M}'$ can be obtained from $\mathcal{M}$ by replacing every symbol $\alpha \in \Sigma^{\mathcal{M}}$ by $\kappa(\alpha) \in \Sigma^{\mathcal{M}'}$. That is if $\mathcal{M}(i,j) = \alpha_1 + \cdots + \alpha_n$, then $\mathcal{M}'(i,j) = \kappa(\alpha_1) + \cdots + \kappa(\alpha_n)$. We write this situation as $\mathcal{M} \overset{\kappa}{\cong} \mathcal{M}'$ (see [28]).

For a symbolic matrix $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ over $\Sigma^{\mathcal{M}}$, we set for $\alpha \in \Sigma^{\mathcal{M}}, i,j = 1,\dots,N$

$$
A^{\mathcal{M}}(i,\alpha,j) = \begin{cases} 1 & \text{if } \alpha \text{ appears in } \mathcal{M}(i,j),\\ 0 & \text{otherwise.} \end{cases} \tag{9.1}
$$

Put an $N\times N$ nonnegative matrix $A^{\mathcal{M}} = [A_{\mathcal{M}}(i,j)]_{i,j=1}^N$ by setting $A^{\mathcal{M}}(i,j) = \sum_{\alpha\in\Sigma^{\mathcal{M}}} A^{\mathcal{M}}(i,\alpha,j)$. Let $\mathcal{A}$ be an N-dimensional commutative C^*-algebra $\mathbb{C}^N$ with minimal projections $E_1,\dots,E_N$ such that

$$
\mathcal{A} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_N.
$$

We set for $\alpha \in \Sigma^{\mathcal{M}}$:

$$
\rho^{\mathcal{M}}_\alpha(E_i) = \sum_{j=1}^N A^{\mathcal{M}}(i,\alpha,j)E_j, \qquad i = 1,\dots,N.
$$

Then we have a C^*-symbolic dynamical system $(\mathcal{A}, \rho^{\mathcal{M}}, \Sigma^{\mathcal{M}})$.

Let $\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$ and $\mathcal{N} = [\mathcal{N}(i,j)]_{i,j=1}^N$ be $N \times N$ symbolic matrices over $\Sigma^{\mathcal{M}}$ and $\Sigma^{\mathcal{N}}$ respectively. We have two C^*-symbolic dynamical systems $(\mathcal{A}, \rho^{\mathcal{M}}, \Sigma^{\mathcal{M}})$ and $(\mathcal{A}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{N}})$. Put

$$\Sigma^{\mathcal{M}\mathcal{N}} = \{(\alpha, b) \in \Sigma^{\mathcal{M}} \times \Sigma^{\mathcal{N}} \mid \rho_b^{\mathcal{N}} \circ \rho_\alpha^{\mathcal{M}} \neq 0\},$$
$$\Sigma^{\mathcal{N}\mathcal{M}} = \{(a, \beta) \in \Sigma^{\mathcal{N}} \times \Sigma^{\mathcal{M}} \mid \rho_\beta^{\mathcal{M}} \circ \rho_a^{\mathcal{N}} \neq 0\}.$$

Suppose that there is a bijection κ from $\Sigma^{\mathcal{M}\mathcal{N}}$ to $\Sigma^{\mathcal{N}\mathcal{M}}$ such that κ yields a specified equivalence

$$\mathcal{M}\mathcal{N} \overset{\kappa}{\cong} \mathcal{N}\mathcal{M} \tag{9.2}$$

and fix it.

Proposition 9.1. *Keep the above situations. The specified equivalence (8.2) induces a specification $\kappa : \Sigma^{\mathcal{M}\mathcal{N}} \longrightarrow \Sigma^{\mathcal{N}\mathcal{M}}$ such that*

$$\rho_b^{\mathcal{N}} \circ \rho_\alpha^{\mathcal{M}} = \rho_\beta^{\mathcal{M}} \circ \rho_a^{\mathcal{N}} \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{9.3}$$

Hence $(\mathcal{A}, \rho^{\mathcal{M}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{M}}, \Sigma^{\mathcal{N}}, \kappa)$ gives rise to a C^*-textile dynamical system.

Proof. Since $\mathcal{M}\mathcal{N} \overset{\kappa}{\cong} \mathcal{N}\mathcal{M}$, one sees that for $i, j = 1, 2, \dots, N$, $\kappa(\mathcal{M}\mathcal{N}(i,j)) = \mathcal{N}\mathcal{M}(i,j)$. For $(\alpha, b) \in \Sigma^{\mathcal{M}\mathcal{N}}$, there exists $i, k = 1, 2, \dots, N$ such that $\rho_b^{\mathcal{N}} \circ \rho_\alpha^{\mathcal{M}}(E_i) \geq E_k$. As $\kappa(\alpha, b)$ appears in $\mathcal{N}\mathcal{M}(i,k)$, by putting $(a, \beta) = \kappa(\alpha, b)$, we have $\rho_\beta^{\mathcal{M}} \circ \rho_a^{\mathcal{N}}(E_i) \geq E_k$. Hence $\kappa(\alpha, b) \in \Sigma^{\mathcal{N}\mathcal{M}}$. One indeed sees that $\rho_b^{\mathcal{N}} \circ \rho_\alpha^{\mathcal{M}} = \rho_\beta^{\mathcal{M}} \circ \rho_a^{\mathcal{N}}$ by the relation $\mathcal{M}\mathcal{N} \overset{\kappa}{\cong} \mathcal{N}\mathcal{M}$. □

Two symbolic matrices satisfying the relations (9.2) give rise to LR textile systems that have been introduced by Nasu (see [28]). Textile systems introduced by Nasu play an important tool to analyze automorphisms and endomorphisms of topological Markov shifts. The author has generalized LR-textile systems to LR-textile λ-graph systems which consist of two pairs of sequences $(\mathcal{M}, I) = (\mathcal{M}_{l,l+1}, I_{l,l+1})_{l \in \mathbb{Z}_+}$ and $(\mathcal{N}, I) = (\mathcal{N}_{l,l+1}, I_{l,l+1})_{l \in \mathbb{Z}_+}$ such that

$$\mathcal{M}_{l,l+1}\mathcal{N}_{l+1,l+2} \overset{\kappa}{\cong} \mathcal{N}_{l,l+1}\mathcal{M}_{l+1,l+2}, \qquad l \in \mathbb{Z}_+ \tag{9.4}$$

through a specification κ ([24]). We denote i the LR-textile λ-graph system by $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$. Denote by $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ the associated λ-graph systems respectively. Since $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ have common sequences $V_l^{\mathcal{M}} = V_l^{\mathcal{N}}, l \in \mathbb{Z}_+$ of vertices which denoted by $V_l, l \in \mathbb{Z}_+$, and its common inclusion matrices $I_{l,l+1}, l \in \mathbb{Z}_+$. Hence $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ form square in the sense of [24, p.170]. Let $(\mathcal{A}_{\mathcal{M}}, \rho^{\mathcal{M}}, \Sigma^{\mathcal{M}})$ and $(\mathcal{A}_{\mathcal{N}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{N}})$ be the associated C^*-symbolic dynamical systems with the λ-graph systems $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ respectively. Since they have common vertices $V_l, l \in \mathbb{Z}_+$ and inclusion matrices $I_{l,l+1}, l \in \mathbb{Z}_+$, the algebras $\mathcal{A}_{\mathcal{M}}$ and $\mathcal{A}_{\mathcal{N}}$ are identical, which is denoted by $\mathcal{A}$. It is easy to seee that the relation (9.4) implies

$$\rho_\alpha^{\mathcal{M}} \circ \rho_b^{\mathcal{N}} = \rho_a^{\mathcal{N}} \circ \rho_\beta^{\mathcal{M}} \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{9.5}$$

Proposition 9.2. *An LR-textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ yields a C^*-textile dynamical system $(\mathcal{A}, \rho^{\mathcal{M}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{M}}, \Sigma^{\mathcal{N}}, \kappa)$ which forms square.*

Conversely, a C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which forms square yields an LR-textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}^\rho}_{\mathcal{M}^\eta}}$ such that the associated C^*-textile dynamical system $(\mathcal{A}_{\rho,\eta}, \rho^{\mathcal{M}^\rho}, \rho^{\mathcal{M}^\eta}, \Sigma^{\mathcal{M}^\rho}, \Sigma^{\mathcal{M}^\rho}, \kappa)$ is a subsystem of $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$

in the sense that the relations:

$$\mathcal{A}_{\rho,\eta} \subset \mathcal{A}, \qquad \rho|_{\mathcal{A}_{\rho,\eta}} = \rho^{\mathcal{M}^\rho}, \qquad \eta|_{\mathcal{A}_{\rho,\eta}} = \rho^{\mathcal{M}^\eta}$$

hold.

Proof. Let $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ be an LR-textile λ-graph system. As in the above discussions, we have a C^*-textile dynamical system $(\mathcal{A}, \rho^{\mathcal{M}}, \rho^{\mathcal{N}}, \Sigma^{\mathcal{M}}, \Sigma^{\mathcal{N}}, \kappa)$. Conversely, let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^*-textile dynamical system which forms square. Put for $l \in \mathbb{N}$

$$\mathcal{A}^\rho_l = C^*(\rho_\mu(1) : \mu \in B_l(\Lambda_\rho)), \qquad \mathcal{A}^\eta_l = C^*(\eta_\xi(1) : \xi \in B_l(\Lambda_\eta)).$$

Since $\mathcal{A}^\rho_l = \mathcal{A}^\eta_l$ and they are commutative and of finite dimensional, the algebra

$$\mathcal{A}_{\rho,\eta} = \overline{\cup_{l\in\mathbb{Z}_+}\mathcal{A}^\rho_l} = \overline{\cup_{l\in\mathbb{Z}_+}\mathcal{A}^\eta_l}$$

is a commutative AF-subalgebra of $\mathcal{A}$. It is easy to see that both $(\mathcal{A}_{\rho,\eta}, \rho, \Sigma^\rho)$ and $(\mathcal{A}_{\rho,\eta}, \eta, \Sigma^\eta)$ are C^*-symbolic dynamical systems such that

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta) \tag{9.6}$$

By MaCrelle, there exist λ-graph systems $\mathfrak{L}^\rho$ and $\mathfrak{L}^\eta$ whose C^*-symbolic dynamical systems are $(\mathcal{A}_{\rho,\eta}, \rho, \Sigma^\rho)$ and $(\mathcal{A}_{\rho,\eta}, \eta, \Sigma^\eta)$ respectively. Let $(\mathcal{M}^\rho, I^\rho)$ and $(\mathcal{M}^\eta, I^\eta)$ be the associated symbolic dynamical systems. It is easy to see that the relation (9.6) implies

$$\mathcal{M}^\rho_{l,l+1}\mathcal{M}^\eta_{l+1,l+2} \overset{\kappa}{\cong} \mathcal{M}^\eta_{l,l+1}\mathcal{M}^\rho_{l+1,l+2}, \qquad l \in \mathbb{Z}_+. \tag{9.7}$$

Hence we have an LR-textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}^\rho}_{\mathcal{M}^\eta}}$. It is direct to see that the associated C^*-textile dynamical system is

$$(\mathcal{A}_{\rho,\eta}, \rho|_{\mathcal{A}_{\rho,\eta}}, \eta|_{\mathcal{A}_{\rho,\eta}}, \Sigma^\rho, \Sigma^\eta, \kappa).$$

□

Let A be an $N \times N$ matrix with entries in nonnegative integers. We may consider a directed graph $G_A = (V_A, E_A)$ with vertex set V_A and edge set E_A. The vertex set V_A consists of N vertices which we denote by $\{v_1, \dots, v_N\}$. We equip $A(i,j)$ edges from the vertex v_i to the vertex v_j. Denote by E_A the set of the edges. Let $\Sigma^A = E_A$ and the labeling map $\lambda_A : E_A \longrightarrow \Sigma^A$ be defined as the identity map. Then we have a labeled directed graph denoted by G_A as well as a symbolic matrix $\mathcal{M}_A = [\mathcal{M}_A(i,j)]_{i,j=1}^N$ by setting

$$\mathcal{M}_A(i,j) = \begin{cases} e_1 + \cdots + e_n & \text{if } e_1, \cdots, e_n \text{ are edges from } v_i \text{ to } v_j, \\ 0 & \text{if there is no edge from } v_i \text{ to } v_j. \end{cases}$$

Let B be an $N \times N$ matrix with entries in nonnegative integers such that

$$AB = BA. \tag{9.8}$$

The equality (9.8) implies that the cardinal numbers of the sets of the pairs of directed edges

$$\{(e,f) \in E_A \times E_B \mid s(e) = v_i, t(e) = s(f), t(f) = v_j\} \text{ and}$$
$$\{(f,e) \in E_B \times E_A \mid s(f) = v_i, t(f) = s(e), t(e) = v_j\}$$

coincide with each other for each v_i and v_j, so that one may take a bijection $\kappa : \Sigma^{AB} \longrightarrow \Sigma^{BA}$ which gives rise to a specified equivalence $\mathcal{M}_A\mathcal{M}_B \overset{\kappa}{\cong} \mathcal{M}_A\mathcal{M}_B$. We then have a C^*-textile dynamical system

$$(\mathcal{A}, \rho^{\mathcal{M}_A}, \rho^{\mathcal{M}_B}, \Sigma^A, \Sigma^B, \kappa)$$

which we denote by

$$(\mathcal{A}, \rho^A, \rho^B, \Sigma^A, \Sigma^B, \kappa).$$

The associated C^*-algebra is denoted by $\mathcal{O}^\kappa_{A,B}$. The algebra $\mathcal{O}^\kappa_{A,B}$ depends on the choice of a specification $\kappa : \Sigma^{AB} \longrightarrow \Sigma^{BA}$. The algebras are 2-graph algebras of Kumjian and Pask [15]. They are C^*-algebras associated to textile systems studied by V. Deaconu [8]. By Theorem 8.23, we have

Proposition 9.3. *Keep the above situations. There exist short exact sequences:*

$$\begin{aligned} 0 &\longrightarrow \mathbb{Z}^N/((1-A)\mathbb{Z}^N + (1-B)\mathbb{Z}^N) \\ &\longrightarrow K_0(\mathcal{O}^\kappa_{A,B}) \\ &\longrightarrow \operatorname{Ker}(1-A) \cap \operatorname{Ker}(1-B) \ \textit{in } \mathbb{Z}^N \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\longrightarrow (\operatorname{Ker}(1-B) \ \textit{in } \mathbb{Z}^N)/(1-A)(\operatorname{Ker}(1-B) \ \textit{in } \mathbb{Z}^N) \\ &\longrightarrow K_1(\mathcal{O}^\kappa_{A,B}) \\ &\longrightarrow \operatorname{Ker}(1-A) \ \textit{in} \quad \mathbb{Z}^N/(1-B)\mathbb{Z}^N \longrightarrow 0. \end{aligned}$$

We consider 1×1 matrices $[N]$ and $[M]$ with its entries N and M respectively for $1 < N, M \in \mathbb{N}$. Let G_N be a directed graph with one vertex and N directed self-loops. Similarly we consider a directed graph G_M with M directed self-loops at the vertex. Denote by $\Sigma^N = \{f_1, \dots, f_N\}$ the set of directed N-self loops of G_N and $\Sigma^M = \{e_1, \dots, e_M\}$ the set of directed M-self loops of G_M. As a specification κ, we take the exchanging map $(e, f) \in \Sigma^M \times \Sigma^N \longrightarrow (f, e) \in \Sigma^N \times \Sigma^M$ which we will fix. Put

$$\rho^M_{e_i}(1) = 1, \qquad \rho^N_{f_j}(1) = 1.$$

Then we have a C^*-textile dynamical system

$$(\mathbb{C}, \rho^M, \rho^N, \Sigma^M, \Sigma^N, \kappa).$$

The associated C^*-algebra is denoted by $\mathcal{O}^\kappa_{M,N}$.

Lemma 9.4. $\mathcal{O}^\kappa_{N,M} = \mathcal{O}_N \otimes \mathcal{O}_M$.

Proof. Let $s_i, i = 1, \dots, N$ and $t_j, i = 1, \dots, M$ be the generating isometries of the Cuntz algebra $\mathcal{O}_N$ and those of of $\mathcal{O}_M$ respectively which satisfy

$$\sum_{i=1}^N s_i s_i^* = 1, \qquad \sum_{j=1}^M t_j t_j^* = 1.$$

Let $S_i, i = 1, \dots, N$ and $T_j, i = 1, \dots, M$ be the generating isometries of $\mathcal{O}^\kappa_{N,M}$ satisfying

$$\sum_{i=1}^N S_i S_i^* = 1, \qquad \sum_{j=1}^M T_j T_j^* = 1$$

and

$$S_i T_j = T_j S_i, \qquad i = 1, \dots, N, \quad j = 1, \dots, M.$$

By the universality of $\mathcal{O}^{\kappa}_{N,M}$ subject to the relations, one has a surjective homomorphism $\Phi : \mathcal{O}_{N,M} \longrightarrow \mathcal{O}_N \otimes \mathcal{O}_M$ such that $\Phi(S_i) = s_i \otimes 1, \ \Phi(T_j) = 1 \otimes t_j$. And also the universality of the tensor product $\mathcal{O}_N \otimes \mathcal{O}_M$, gives rise to its inverse so that Φ is isomorphic. $\square$

Although we may easily compute the K-groups $K_*(\mathcal{O}^{\kappa}_{M,N})$ by using the Künneth formula for $K_i(\mathcal{O}_N \otimes \mathcal{O}_M)$ ([40]), we will compute them by Proposition 9.3 as in the following way.

Proposition 9.5 (cf.[15]). *For $1 < N, M \in \mathbb{N}$, the C^*-algebra $\mathcal{O}^{\kappa}_{N,M}$ is simple, purely infinite, such that*

$$K_0(\mathcal{O}^{\kappa}_{N,M}) \cong K_1(\mathcal{O}^{\kappa}_{N,M}) \cong \mathbb{Z}/d\mathbb{Z}$$

where $d = g.c.d(N-1, M-1)$ the greatest common diviser of $N-1, M-1$.

Proof. It is easy to see that the group $\mathbb{Z}/((N-1)\mathbb{Z} + (N-1)\mathbb{Z})$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$. As $\mathrm{Ker}(N-1) = \mathrm{Ker}(M-1) = 0$ in $\mathbb{Z}$, we see that $K_0(\mathcal{O}^{\kappa}_{N,M}) \cong \mathbb{Z}/d\mathbb{Z}$.

It is elementary to see that the subgroup

$$\{[k] \in \mathbb{Z}/(M-1)\mathbb{Z} \mid (N-1)k \in (M-1)\mathbb{Z}\}$$

of $\mathbb{Z}/(M-1)\mathbb{Z}$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$. Hence we have $K_1(\mathcal{O}^{\kappa}_{N,M}) \cong \mathbb{Z}/d\mathbb{Z}$. $\square$

We will generalize the above examples from the view point of tensor products.

2. Tensor products.

Let $(\mathcal{A}^{\rho}, \rho, \Sigma^{\rho})$ and $(\mathcal{A}^{\eta}, \eta, \Sigma^{\eta})$ be C^*-symbolic dynamical systems. We will construct a C^*-textile dynamical system by taking tensor products. Put

$$\bar{\mathcal{A}} = \mathcal{A}^{\rho} \otimes \mathcal{A}^{\eta}, \qquad \bar{\rho}_{\alpha} = \rho_{\alpha} \otimes \mathrm{id}, \qquad \bar{\eta}_a = \mathrm{id} \otimes \eta_a, \qquad \Sigma^{\bar{\rho}} = \Sigma^{\rho}, \qquad \Sigma^{\bar{\eta}} = \Sigma^{\eta}$$

for $\alpha \in \Sigma^{\rho}, a \in \Sigma^{\eta}$, where $\otimes$ means the minimal C^*-tensor product $\otimes_{\min}$. For $(\alpha, a) \in \Sigma^{\rho} \times \Sigma^{\eta}$, we see $\eta_b \circ \rho_{\alpha}(1) \neq 0$ if and only if $\eta_b(1) \neq 0, \rho_{\alpha}(1) \neq 0$, so that

$$\Sigma^{\bar{\rho}\bar{\eta}} = \Sigma^{\rho} \times \Sigma^{\eta} \quad \text{and similarly} \quad \Sigma^{\bar{\eta}\bar{\rho}} = \Sigma^{\eta} \times \Sigma^{\rho}.$$

Define $\bar{\kappa} : \Sigma^{\bar{\rho}\bar{\eta}} \longrightarrow \Sigma^{\bar{\eta}\bar{\rho}}$ by setting $\bar{\kappa}(\alpha, b) = (b, \alpha)$.

Lemma 9.6. *$(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is a C^*-textile dynamical system.*

Proof. By [1], we have $Z_{\bar{\mathcal{A}}} = Z_{\mathcal{A}^{\rho}} \otimes Z_{\mathcal{A}^{\eta}}$ so that

$$\bar{\rho}_{\alpha}(Z_{\bar{\mathcal{A}}}) \subset Z_{\bar{\mathcal{A}}}, \quad \alpha \in \Sigma^{\bar{\rho}} \quad \text{and} \quad \bar{\rho}_a(Z_{\bar{\mathcal{A}}}) \subset Z_{\bar{\mathcal{A}}}, \quad a \in \Sigma^{\bar{\eta}}.$$

We also have $\sum_{\alpha \in \Sigma^{\bar{\rho}}} \bar{\rho}_{\alpha}(1) = \sum_{\alpha \in \Sigma^{\rho}} \rho_{\alpha}(1) \otimes 1 \geq 1$, and similarly $\sum_{a \in \Sigma^{\bar{\eta}}} \bar{\eta}_(1) \geq 1$ so that both families $\{\bar{\rho}_{\alpha}\}_{\alpha \in \Sigma^{\bar{\rho}}}$ and $\{\bar{\eta}_a\}_{a \in \Sigma^{\bar{\eta}}}$ of endomorphisms are essential. Since $\{\rho_{\alpha}\}_{\alpha \in \Sigma^{\rho}}$ is faithful on $\mathcal{A}^{\rho}$, the homomorphism

$$x \in \mathcal{A}^{\rho} \longrightarrow \oplus_{\alpha \in \Sigma^{\rho}} \rho_{\alpha}(x) \in \oplus_{\alpha \in \Sigma^{\rho}} \mathcal{A}^{\rho}$$

is injective so that the homomorphism

$$x \otimes y \in \mathcal{A}^{\rho} \otimes \mathcal{A}^{\eta} \longrightarrow \oplus_{\alpha \in \Sigma^{\rho}} \rho_{\alpha}(x) \otimes y \in \oplus_{\alpha \in \Sigma^{\rho}} \mathcal{A}^{\rho} \otimes \mathcal{A}^{\eta}$$

is injective. This implies that $\{\bar{\rho}_{\alpha}\}_{\alpha \in \Sigma^{\bar{\rho}}}$ is faithful and similary so is $\{\bar{\eta}_a\}_{a \in \Sigma^{\bar{\eta}}}$. Hence $(\bar{\mathcal{A}}, \bar{\rho}, \Sigma^{\bar{\rho}})$ and $(\bar{\mathcal{A}}, \bar{\eta}, \Sigma^{\bar{\eta}})$ are C^*-symbolic dynamical systems. It is direct to see that $\bar{\eta}_b \circ \bar{\rho}_{\alpha} = \bar{\rho}_{\alpha} \circ \bar{\eta}_b$ for $(\alpha, b) \in \Sigma^{\bar{\rho}\bar{\eta}}$. Therefore $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is a C^*-textile dynamical system. $\square$

We call $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ the tensor product between $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$. Denote by $S_\alpha, \alpha \in \Sigma^{\bar{\rho}}, T_a, a \in \Sigma^{\bar{\eta}}$ the generating partial isometries of the C^*-algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ for the C^*-textile dynamical system $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$. By the universality for the algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ subject to the relations $(\bar{\rho}, \bar{\eta}; \bar{\kappa})$, the algebra $\mathcal{D}_{\bar{\rho},\bar{\eta}}$ is isomorphic to the tensor product $\mathcal{D}_\rho \otimes \mathcal{D}_\eta$ through the correspondence

$$S_\mu T_\xi (x \otimes y) T_\xi^* S_\mu^* \longleftrightarrow S_\mu x S_\mu^* \otimes T_\xi y T_\xi^*$$

for $\mu \in B_*(\Lambda_\rho), \xi \in B_*(\Lambda_\eta), \ x \in A^\rho, y \in \mathcal{A}^\eta$.

Lemma 9.7. *Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free (resp. AF-free). Then the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is free (resp. AF-free).*

Proof. Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free. There exist increasing sequences $\mathcal{A}^\rho_l, l \in \mathbb{Z}_+$ and $\mathcal{A}^\eta_l, l \in \mathbb{Z}_+$ of C^*-subalgebras of $\mathcal{A}^\rho$ and $\mathcal{A}^\eta$ satisfying the conditions of the freeness respectively. Put $\bar{\mathcal{A}}_l = \mathcal{A}^\rho_l \otimes \mathcal{A}^\eta_l, l \in \mathbb{Z}_+$ It is clear that

(1) $\bar{\rho}_\alpha(\bar{\mathcal{A}}_l) \subset \bar{\mathcal{A}}_{l+1}, \alpha \in \Sigma^{\bar{\rho}}$ and $\bar{\eta}_a(\bar{\mathcal{A}}_l) \subset \bar{\mathcal{A}}_{l+1}, a \in \Sigma^{\bar{\eta}}$ for $l \in \mathbb{Z}_+$.
(2) $\cup_{l \in \mathbb{Z}_+} \bar{\mathcal{A}}_l$ is dense in $\bar{\mathcal{A}}$.

We will show that the condition (3) in the definition of freeness holds. Take and fix arbitrary $j, k, l \in \mathbb{N}$ with $j + k \le l$. For $j \le l$, by the freeness of $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ there exists a projection $q_\rho \in \mathcal{D}_\rho \cap \mathcal{A}^{\rho\,\prime}_l$ such that

(i) $q_\rho x \ne 0$ for $0 \ne x \in \mathcal{A}^\rho_l$,
(ii) $\phi^n_\rho(q_\rho) q_\rho = 0$ for all $n = 1, 2, \dots, j$.

Similarly for $k \le l$, by the freeness of $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ there exists a projection $q_\eta \in \mathcal{D}_\eta \cap \mathcal{A}^{\eta\,\prime}_l$ such that

(i) $q_\eta y \ne 0$ for $0 \ne y \in \mathcal{A}^\eta_l$,
(ii) $\phi^m_\eta(q_\eta) q_\eta = 0$ for all $m = 1, 2, \dots, k$.

Put $q = q_\rho \otimes q_\eta \in \mathcal{D}_\rho \otimes \mathcal{D}_\eta (= \mathcal{D}_{\bar{\rho},\bar{\eta}})$ so that $q \in \mathcal{D}_{\bar{\rho},\bar{\eta}} \cap \bar{\mathcal{A}}'_l$. As the maps $\Phi^\rho_l : x \in \mathcal{A}^\rho_l \longrightarrow q_\rho x \in q_\rho \mathcal{A}^\rho_l$ and $\Phi^\eta_l : y \in \mathcal{A}^\eta_l \longrightarrow q_\eta x \in q_\eta \mathcal{A}^\eta_l$ are isomorphisms, the tensor product

$$\Phi^\rho_l \otimes \Phi^\eta_l : x \otimes y \in \mathcal{A}^\rho_l \otimes \mathcal{A}^\eta_l \longrightarrow (q_\rho \otimes q_\eta)(x \otimes y) \in (q_\rho \otimes q_\eta)(\mathcal{A}^\rho_l \otimes \mathcal{A}^\eta_l)$$

is isomorphic. Hence $qa \ne 0$ for $0 \ne a \in \bar{\mathcal{A}}_l$. It is straightforward to see that $\phi^n_\rho(q)\phi^m_\eta(q) = \phi^n_\rho((\phi^m_\eta(q)))q = \phi^n_\rho(q)q = \phi^m_\eta(q)q = 0$ for all $n = 1, 2, \dots, j$, $m = 1, 2, \dots, k$. Therefore the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is free. It is obvious to see that if both $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are AF-free, then $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is AF-free. $\square$

Proposition 9.8. *Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free. Then the C^*-algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ for the tensor product C^*-textile dynamical system $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is isomorphic to the tensor product $\mathcal{O}_\rho \otimes \mathcal{O}_\eta$ of the C^*-algebras between $\mathcal{O}_\rho$ and $\mathcal{O}_\eta$. If in particular, $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both irreducible, the C^*-algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ is simple.*

Proof. Suppose that $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both free. By the preceding lemma, the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ is free and hence satisfies condition (I). Let $s_\alpha, \alpha \in \Sigma^\rho$ and $t_a, a \in \Sigma^\eta$ be the generating partial isometries of the C^*-algebras $\mathcal{O}_\rho$ and $\mathcal{O}_\eta$ respectively. Let $S_\alpha, \alpha \in \Sigma^{\bar{\rho}}$ and $T_a, a \in \Sigma^{\bar{\eta}}$ be the generating

partial isometries of the C^*-algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$. By the uniqueness of the algebra $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ with respect to the relations $(\bar{\rho},\bar{\eta};\bar{\kappa})$, the correspondence

$$S_\alpha \longrightarrow s_\alpha \otimes 1 \in \mathcal{O}_\rho \otimes \mathcal{O}_\eta, \qquad T_a \longrightarrow 1 \otimes t_a \in \mathcal{O}_\rho \otimes \mathcal{O}_\eta$$

naturally gives rise to an isomorphism from $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ onto the tensor product $\mathcal{O}_\rho \otimes \mathcal{O}_\eta$.

If in particular, $(\mathcal{A}^\rho, \rho, \Sigma^\rho)$ and $(\mathcal{A}^\eta, \eta, \Sigma^\eta)$ are both irreducible, the C^*-algebras $\mathcal{O}_\rho$ and $\mathcal{O}_\eta$ are both simple so that $\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}}$ is simple. □

We remark that the tensor product $(\bar{\mathcal{A}}, \bar{\rho}, \bar{\eta}, \Sigma^{\bar{\rho}}, \Sigma^{\bar{\eta}}, \bar{\kappa})$ does not necessarily form square. The K-theory groups $K_*(\mathcal{O}^{\bar{\kappa}}_{\bar{\rho},\bar{\eta}})$ are computed from the Künneth formulae for $K_*(\mathcal{O}_\rho \otimes \mathcal{O}_\eta)$ [40].

In [27], higher dimensional analogue $(\mathcal{A}, \rho^1, \dots, \rho^N, \Sigma^1, \dots, \Sigma^N, \kappa)$ will be studied.

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Department of Mathematics, Joetsu University of Education, Joetsu 943-8512, Japan
E-mail address: `kengo@juen.ac.jp`